Convergent expansions for Random Cluster Model with $q > 0$ on infinite graphs

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February 8, 2008

Abstract

In this paper we extend our previous results on the connectivity functions and pressure of the Random Cluster Model in the highly subcritical phase and in the highly supercritical phase, originally proved only on the cubic lattice $\mathbb{Z}^d$, to a much wider class of infinite graphs. In particular, concerning the subcritical regime, we show that the connectivity functions are analytic and decay exponentially in any bounded degree graph. In the supercritical phase, we are able to prove the analyticity of finite connectivity functions in a smaller class of graphs, namely, bounded degree graphs with the so called minimal cut-set property and satisfying a (very mild) isoperimetric inequality. On the other hand we show that the large distances decay of finite connectivity in the supercritical regime can be polynomially slow depending on the topological structure of the graph. Analogous analyticity results are obtained for the pressure of the Random Cluster Model on an infinite graph, but with the further assumptions of amenability and quasi-transitivity of the graph.

1 Introduction

In recent years there has been an increasing interest about statistical mechanics systems and stochastic processes on general infinite graphs. The main motivation has been the possible connections and applications in computer science, with particular attention to reliability of large network (e.g. the internet). More recently, see e.g. [38, 39, 35, 6, 11], people started to realize that ideas and methods of statistical mechanics could be useful to answer questions arising in combinatorics and graph theory.

Rigorous results on this subject have appeared since the early nineties and nowadays there is a consistent literature on this subject. Actually, the study of statistical mechanics and percolation processes on infinite graphs other than the usual unit cubic lattice $\mathbb{Z}^d$ or planar triangular and hexagonal lattices has been limited essentially to non amenable graphs. Roughly speaking, the non amenable graphs are those for which the ratio of the boundary of the graph and its interior does not go to zero in the infinite volume limit, while for amenable graphs this ratio goes to zero. Within the class of non amenable graphs, the study has been mostly focused on trees (i.e.,

graphs with no circuits) , see e.g. [4, 5, 23, 20, 21, 22, 37, 25, 26, 27, 19, 18]. There have been also a few papers dealing with percolation processes on quasi transitive or transitive graphs, including amenable graphs, see e.g. [2], [3], [30]. Roughly speaking, in a transitive graph G any vertex of the graph is equivalent; in other words G "looks the same" by observers sitting in different vertices. In a quasi-transitive graph G there is a finite number of different types of vertices and G "looks the same" by observers sitting in vertices of the same type.

Some general results about percolation on general infinite graphs (i.e. not necessarily non amenable and/or quasi-transitive) appeared in [4], [5], and [2], and more recently in [33] (see also references therein). There are also some other works about the Potts model, in particular the antiferromagnetic case, on general finite graphs [38, 39] and on amenable quasi-transitive infinite graphs [35].

In this paper we focus our attention on the study of the dependent percolation process known as Random Cluster Model (RCM) on general infinite graphs.

The RCM was proposed by Fortuin and Kasteleyn in the early seventies [12] as a generalization of the Potts model. The RCM on a graph G depends on two real parameters: the parameter $p \in [0, 1]$ and the parameter $q \in (0, +\infty)$. The parameter p represents the weight of an edge of G to be open independently of the other edges and it is related to the temperature of the Potts model. The parameter q, when different from 1, introduces a dependence in the percolation process described by RCM, and, when integer greater than 2, it represents the number of colors in the Potts model.

Some results on RCM can be proved for all the values of the parameters q and p. In particular, there are results about the logarithm of the total weight of the measure (pressure). Namely, the existence of the pressure of the RCM, its independency on boundary conditions and its differentiability (with respect to p almost everywhere in the interval $[0, 1]$) have been proved for all $q \in (0, \infty)$ when the underlying infinite graph is the cubic lattice $\mathbb{Z}^d$ in [15] (see also [23] for some generalization of such results to transitive amenable graphs). This shows that in these cases the whole machinery of the statistical mechanics, and its probabilistic counterpart, can be used for all the values of the parameters of the RCM.

However, the study of the statistical mechanics properties of RCM has been developed so far mainly in $\mathbb{Z}^d$, and only in the region $q \geq 1$, where the powerful tool given by the so-called FKG inequalities is availabe. In particular, by comparison inequalities (see [12], [1] and [16]), it is possible to prove that the RCM on $\mathbb{Z}^d$ admits, for $q \geq 1$, a (non trivial) critical value $p_c(q) \in (0, 1)$ such that for $p < p_c(q)$ the probability to have an infinite open cluster is zero, while for $p > p_c(q)$ is one ([1], Theorem 4.2). Many other important results can be collected for the RCM on $\mathbb{Z}^d$ in the regime $q \geq 1$. We refer the reader to the monograph [16] and book [17] for a detailed description of these results and references.

Concerning the case $q < 1$, due to the lack of validity of FKG inequalities in this regime, nearly quoting the words of Grimmett in [16], many fundamental questions are unanswered to date, and the theory of RCM remains obscure when $q < 1$. We tried to answer to some of these questions in a recent paper [34], where we studied, by mean of cluster expansion methods, the statistical mechanics behavior of the Random Cluster Model on the cubic lattice $\mathbb{Z}^d$ ($d \geq 2$) for p near either 0 or 1 and for all $q > 0$, proving the analyticity of the pressure and of finite connectivities in both regimes. The results of [34] also give a generalization of theorem 4.2. in [1] for values of q in the interval $0 < q < 1$.

In the present paper, by taking advantage of the robustness and malleability of cluster expansion methods, we continue the analysis of the statistical mechanics behavior of the RCM, and in particular its analyticity properties for p near either 0 or 1, extending the results of

[34] to RCM on a class of graph much more general than the regular lattices like $\mathbb{Z}^d$. Here we are motivated by recent results [38], [39] [35], [33] showing how statistical mechanics techniques (and in particular cluster expansion) may give interesting contributions to specific problems concerning graph theory.

Our results are stated in a detailed form in theorems 4.1, 4.2, 5.9, 5.13. However, for the benefit of the readers, we report sketchily these results here below.

For the subcritical regime we obtain that, for any fixed value of $q > 0$ there is $R_q^{sub} > 0$ such that for any p in the disk $|p| \leq R_q^{sub}$ we have the following results:

1a) The n-point connectivity functions ($n \in \mathbb{N}$) of the RCM on an infinite graph $\mathbb{G}$ can be written explicitly as analytic functions of p whenever $\mathbb{G}$ is bounded degree and they decay exponentially fast at large distances, which also implies that the probability to have an infinite open cluster in the graph is zero when $p \in [0, R_q^{sub})$. These results have been obtained via a limit procedure on sequences of subgraphs of $\mathbb{G}$, and we are able to prove that the limit of the n point connectivity function tends to the same analytic function for free and wired boundary condition.

1b) The pressure is analytic in p in the same region whenever $\mathbb{G}$ is quasi-transitive and amenable.

For the supercritical regime we obtain that, for any fixed value of $q > 0$ there exists $R_q^{sup} > 0$ such that for any p in the disk $|1 - p| < R_q^{sup}$ we have the following results:

2a) For any $n \geq 1$, the n-point finite connectivity function of the RCM on an infinite graph $\mathbb{G}$ can be written explicitly as an analytic function of $1 - p$ whenever $\mathbb{G}$ is bounded degree and satisfies some additional properties, including a very weak isoperimetric inequality (see below). Such result immediately implies that for any p in the interval $(1 - R_q^{sup}, 1]$ the probability to have an infinite open cluster in the graph containing a fixed vertex is strictly greater than zero.

We remark that the class of graphs for which we can prove analyticity of correlations in the supercritical regime is smaller than the class of bounded degree graphs, but it is still very large class: e.g., it contains $\mathbb{Z}^d$ and all the regular lattices and also graphs without symmetries. This result is obtained with a limit construction on finite subgraphs of $\mathbb{G}$, independently of free or wired boundary conditions. Differently from the subcritical regime, the finite connectivity functions may decay in the supercritical phase with a rate that can be polynomially slow, depending on the topological structure of the graph. We plan to investigate in details this feature of the supercritical phase on general graphs in a forthcoming paper devoted only to Bernoulli percolation (i.e. Random Cluster Model with $q = 1$). Indeed results of this paper suggest that the decay rate of finite connectivities for the Bernoulli percolation process on an infinite graph can be adopted as an efficient and quantitative measure of the degree of connection of the graph. Namely, the more rapid is the decay rate of connectivities, the more dense (or connected) is the graph.

2b) The pressure is analytic in $1 - p$ if $\mathbb{G}$ is in the class above and it is (vertex and edge) quasi-transitive and amenable.

Our conditions on the structure of the graph guaranteeing the convergence of the cluster expansion in the subcritical phase are quite general. In particular, for the existence and convergence of the connectivity functions, it is just required for the graph to be bounded degree, which constitutes a very large class of graphs. However, it is possible that with similar techniques one can study unbounded degree graphs in which the vertices with large degree are "rare enough". The requirement of amenability and quasi-transitivity for the existence of the pressure is also largely expected. Roughly speaking, amenability guarantees the possibility to perform the thermodynamic limit in the Van-Hove sense, so that the effects of the boundary vanish in

the infinite volume limit. Quasi-transitivity plays the role of "translational invariance" in the graph which is in general a necessary tool for the existence of the pressure.

On the other hand, in the supercritical case we think that the conditions above are far from optimal. In particular, the isoperimetric condition is due to technical reasons in view to adapt the Peierls argument and contour theory to general graphs.

The paper is organized as follows. In section 2 we give some definitions about graphs. In section 3 we introduce the model, first on finite graphs and then on infinite graphs. In section 4 we study the highly subcritical phase, and state two theorems (theorem 4.1 and theorem 4.2), the first one concerning the connectivity functions and the second one concerning the pressure. The rest of the section is devoted to the proof of these two theorems. In section 5 we perform the analysis of the supercritical phase. Namely, in subsection 5.1 we give some more definitions and properties about cut sets in infinite graphs and, at the end of the subsection, we state the results on the supercritical phase in form of two more theorems: theorem 5.9 concerns the connectivity functions and theorem 5.13 concerns the pressure. In section 5.2 we construct the polymer expansion for the connectivity functions. In section 5.3 we show that this expansion is absolutely convergent for p sufficiently close to 1 and we conclude the proof of theorem 5.9. In section 5.4 we prove theorem 5.13.

2 Some definitions about graphs

For any finite or countable set V, we will denote by $|V|$ the cardinality of V. We denote by $\mathrm{P}_n(V)$ the set of all subsets $U \subset V$ such that $|U| = n$ and we denote by $\mathrm{P}_{\geq n}(V)$ the set of all *finite* subsets $U \subset V$ such that $n \leq |U| < +\infty$. A *graph* is a pair $G = (V, E)$ with V being a countable set, and $E \subset \mathrm{P}_2(V)$. The elements of V are called *vertices* of G and the elements of E are called *edges* of G. A graph $G = (V, E)$ is *finite* if $|V| < \infty$, and infinite otherwise. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Then $G \cup G' = (V \cup V', E \cup E')$. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G, written as $G' \subseteq G$.
Two vertices x and y of G are *adjacent* if $\{x, y\}$ is an edge of G. The *degree* d_x of a vertex $x \in V$ in G is the number of vertices y adjacent to x. A graph $G = (V, E)$ is *locally finite* if $d_x < +\infty$ for all $x \in V$, and it is *bounded degree, with maximum degree* Δ, if $\max_{x \in V}\{d_x\} \leq \Delta < \infty$. A graph $G = (V, E)$ is *connected* if for any pair B, C of subsets of V such that $B \cup C = V$ and $B \cap C = \emptyset$, there is an edge $e \in E$ such that $e \cap B \neq \emptyset$ and $e \cap C \neq \emptyset$. A graph $G = (V, E)$ is a called a *tree graph* or simply a *tree* if it is connected and $|E| = |V| - 1$.
Hereafter the symbol $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ will denote an *infinite and connected* graph.
A *path* in a graph G is a sub-graph $\tau = (V_\tau, E_\tau)$ of G such that

$$V_\tau = \{x_1, x_2, \ldots, x_n\} \qquad E_\tau = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}\}$$

where all x_i are distinct. The vertices x_1 and x_n are called end-vertices of the path, while the vertices $x_2, \ldots, x_{n-1}$ are called the inner vertices of τ and we say that τ connects (or links) x_1 to x_n, (as well as τ is a path from x_1 to x_n). The length $|\tau|$ of a path $\tau = (V_\tau, E_\tau)$ is the number of its edges, i.e. $|\tau| = |E_\tau|$. A path in G is also called a *self avoiding walk (SAW)* in G.
Given a graph $G = (V, E)$ and two distinct vertices $x, y \in V$, we denote by $\mathcal{P}_G^{xy}$ the set of all paths in G connecting x to y. The *distance* $d_G(x, y)$ between two vertices x, y of G is the number $d_G(x, y) = \min\{|\tau| : \tau \in \mathcal{P}_G^{xy}\}$. Note that $d_G(x, y) = 1$ if and only if $\{x, y\} \in E$. Given two edges e and e' of G, we define $d_G(e, e') = \min\{d_G(x, y) : x \in e, y \in e'\}$. If $S, R \subset V$ then $d_G(S, R) = \min\{d_G(x, y) : x \in S, y \in R\}$. If $F, H \subset E$ then $d_G(F, H) = \min\{d_G(e, e') : e \in F, e' \in H\}$.

Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be an infinite connected graph. A *ray* $\rho = (\mathbb{V}_\rho, \mathbb{E}_\rho)$ in $\mathbb{G}$ is an *infinite* sub-graph of $\mathbb{G}$ such that

$$\mathbb{V}_\rho = \{x_0, x_1, x_2, \dots, x_n, \dots\} \qquad \mathbb{E}_\rho = \{\{x_0, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}, \dots\}$$

where all x_i are distinct. The vertex x_0 is called the *starting vertex* of the ray and we say that ρ *starts at* x_0. We denote by $\mathcal{R}^x_{\mathbb{G}}$ the set of all rays in $\mathbb{G}$ starting at x. A ray $\rho = (\mathbb{V}_\rho, \mathbb{E}_\rho)$ in $\mathbb{G}$ with starting vertex x_0 is *geodesic* if $d_G(x_0, x_n) = n$ for all $x_n \in \mathbb{V}_\rho$.
Let ρ and ρ' be two geodesic rays with the same starting vertex x with vertex sets $\mathbb{V}_\rho = \{x_0 = x, x_1, x_2, \dots, x_n, \dots\}$ and $\mathbb{V}_{\rho'} = \{y_0 = x, y_1, y_2, \dots, y_n, \dots\}$ respectively. If $\mathbb{V}_\rho$ and $\mathbb{V}_{\rho'}$ are such that $d_G(x_n, y_m) = n + m$ for any $\{n, m\} \in \mathbb{N}$, then the union $\delta = \rho \cup \rho'$ is called a *geodesic diameter* (or *bi-infinite geodesic*) in $\mathbb{G}$.
Given $G = (V, E)$ connected and $R \subset V$, let $E|_R = \{\{x, y\} \in E : x \in R, y \in R\}$ and define the graph $G|_R = (R, E|_R)$. Note that $G|_R$ is a sub-graph of G. We call $G|_R$ *the restriction of G to R*. We say that $R \subset V$ is *connected* if $G|_R$ is connected. Analogously, Given $G = (V, E)$ connected and $\eta \subset E$, let $V|_\eta = \{x \in V : x \in e \text{ for some } e \in \eta\}$. We call $V|_\eta$ the *support* of η. We say that a edge set $\eta \in E$ is *connected* if the graph $g = (V|_\eta, \eta)$ is connected.
For any non empty $R \subset V$, we denote by $\partial_e R$ the (edges) boundary of R defined by

$$\partial_e R = \{e \in E - E|_R : |e \cap R| = 1\} \tag{2.1}$$

We also denote by $\partial_v^{\rm ext} R$ the *external vertex boundary* of R the subset of $V \backslash R$ given by

$$\partial_v^{\rm ext} R = \{v \in V \backslash R : \ \ \exists e \in E : e = \{v, v'\} \ \text{ with } \ v' \in V\} \tag{2.2}$$

and we denote by $\partial_v^{\rm int} R$ the *internal vertex boundary* of R the subset of R given by

$$\partial_v^{\rm int} R = \{v \in R : \ \ \exists e \in E : e = \{v, v'\} \ \text{ with } \ v' \in V \backslash R\} \tag{2.3}$$

If $R \subset V$ we denote

$$\mathrm{diam}(R) = \sup_{x, y \in R} d_G(x, y) \tag{2.4}$$

and call it the *diameter* of R.
Let $g = (V_g, E_g)$ be a subset of G then we define ∂g the (edge) boundary of g as

$$\partial g = \{e \in E - E_g : \ e \cap V_g \neq \emptyset\}$$

Note that $\partial(G|_R) = \partial_e R$.
Let $G = (V, E)$ be a graph and let $x \in V$ and $R > 0$. We denote by $B(x, R)$ the ball of radius R and center at x, namely $B(x, R) = \{y \in V : \ d_G(x, y) \le R\}$.

Definition 2.1 . *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be an infinite connected graph and let $X \subset \mathbb{V}$ finite. Let now $\mathcal{T}_X$ denote the set of all trees with vertex set X (we recall that a tree in X is a connected graph $\tau = (V_\tau, E_\tau)$ with $V_\tau = X$ and $|E_\tau| = |X| - 1$). We define the minimal tree distance $d^{\rm tree}_{\mathbb{G}}(X)$ of X in $\mathbb{G}$, as*

$$d^{\rm tree}_{\mathbb{G}}(X) = \min_{\tau \in \mathcal{T}_X} \sum_{\{x, y\} \in E_\tau} d_{\mathbb{G}}(x, y) \tag{2.5}$$

We remark that in this definition X is not necessarily connected in $\mathbb{E}$. So E_τ is not necessarily a subset of $\mathbb{E}|_X$, so the pair $\{x, y\}$ does not, in general, belong to $\mathbb{E}$, and for that pair $d_\mathbb{G}(x, y) > 1$. On the other hand, note that when X is connected in $\mathbb{G}$ then it is always possible to find some tree τ in $\mathcal{T}_X$ such that $d_\mathbb{G}(x, y) = 1$ for any pair $\{x, y\} \in \tau$ and hence in this case $d^{\rm tree}_\mathbb{G}(X) = |X| - 1$.

Definition 2.2 . *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a connected and infinite graph. We define the connective constant $C_\mathbb{G}$, of $\mathbb{G}$ as*

$$C_\mathbb{G} = \sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{V}} [c_x(n)]^{1/n} \tag{2.6}$$

with $c_x(n)$ being the number of all paths (i.e. Self Avoiding Walks) of length n with starting point x. By definition, for any infinite graph $\mathbb{G}$, we have that $C_\mathbb{G} \in [0, +\infty) \cup \{+\infty\}$.

For example, for a regular tree $\mathbb{T}_k$ of degree k, $C_{\mathbb{T}_k} = k$. For $\mathbb{Z}^2$ the connectivity constant is not known exactly but it is known to belongs to the interval $[2, 62, 2, 68]$

An *automorphism* of a graph $G = (V, E)$ is a bijective map $\gamma : V \to V$ such that $\{x, y\} \in E \Rightarrow \{\gamma x, \gamma y\} \in E$. A graph $G = (V, E)$ is called *transitive* if, for any $x, y \in V$, there exists an automorphism γ of G such that $\gamma(x) = y$.

An infinite connected graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is called *vertex quasi-transitive* (*edge quasi-transitive*) if $\mathbb{V}$ ($\mathbb{E}$) can be partitioned in finitely many sets $\mathbb{O}_1, \dots \mathbb{O}_s$ (orbits) such that for $\{x, y\} \in \mathbb{O}_i$ ($\{e, e'\} \in \mathbb{O}_i$) it exists an automorphism γ on $\mathbb{G}$ which maps x to y (e to e') and this holds for all $i = 1, \dots, s$. If $x \in \mathbb{O}_i$ and $y \in \mathbb{O}_i$ ($e \in \mathbb{O}_i$ and $e' \in \mathbb{O}_i$) we say that x and y (x and y) are equivalent.

Roughly speaking in a transitive infinite graph any vertex of the graph is equivalent; in other words $\mathbb{G}$ "looks the same" by observers sitting in different vertices. In a quasi-transitive infinite graph there is a finite number of different type of vertices and $\mathbb{G}$ "looks the same" by observers sitting in vertices of the same type.

As an immediate example all periodic lattices with the elementary cell made by one site (e.g. square lattice, triangular lattice, hexagonal lattice, etc.) are transitive infinite graphs, while periodic lattices with the elementary cell made by more than one site are quasi-transitive infinite graphs.

Definition 2.3 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a connected infinite graph. $\mathbb{G}$ is said to be amenable if*

$$\inf \left\{ \frac{|\partial_e W|}{|W|} : W \subset \mathbb{V},\ 0 < |W| < +\infty \right\} = 0$$

A sequence $\{V_N\}_{N\in\mathbb{N}}$ of finite sub-sets of $\mathbb{V}$ in an amenable graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is called a Følner sequence if

$$\lim_{N\to\infty} \frac{|\partial_e V_N|}{|V_N|} = 0 \tag{2.7}$$

Note that such definition reminds the notion of Van Hove sequence in statistical mechanics.

Definition 2.4 *Let $\mathbb{V}$ be an infinite countable set. We say that a sequence $\{V_N\}_{N\in\mathbb{N}}$ of $\mathbb{V}$ tends monotonically to $\mathbb{V}$, and we write $V_N \nearrow \mathbb{V}$, if, for all $N \in \mathbb{N}$, V_N is connected, $V_N \subset V_{N+1}$, and $\cup_{N\in\mathbb{N}} V_N = \mathbb{V}$.*

Roughly speaking, amenability in an infinite connected graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ means that the boundary of finite connected set $X \subset \mathbb{V}$ grows slower than its interior as soon as $X \nearrow \mathbb{V}$. For example, $\mathbb{Z}^d$ is amenable, while the regular tree $\mathbb{T}_k$ for $k \geq 3$ is not amenable.

Let us denote by $\mathcal{G}$ the class of locally finite infinite connected graphs and by $\mathcal{B}$ the class of bounded degree infinite connected graphs. We further denote by $\mathcal{Q}^v$ ($\mathcal{Q}^e$) the class of vertex (edge) quasi-transitive graphs, and by $\mathcal{A}$ the class of amenable graphs. In this paper we will not consider non locally finite graphs.

3 The Model

We define initially the model on a *finite* graph $G = (V, E)$. For each edge $e \in E$ we define a binary random variable $n(e)$, which can assume the values $n(e) = 1$ (open edge) and $n(e) = 0$ (closed edge). A configuration ω_G of the process is a function $\omega : E \to \{0, 1\} : e \mapsto n(e)$. We call Ω_G the configuration space, i.e. the set of all possible configurations of random variables $n(e)$ at the edges $e \in E$ of the graph G. Given $\omega \in \Omega_G$ we denote by $O(\omega)$ the subset of E given by $O(\omega) = \{e \in E : \omega(e) = 1\}$ and by $C(\omega)$ the set $C(\omega) = \{e \in E : \omega(e) = 0\}$. An *open connected component* g of ω is a connected subgraph $g = (V_g, E_g)$ of G such that $E_g \neq \emptyset$, $\omega(e) = 1$ for all $e \in E_g$, and $\omega(e) = 0$ for all $e \in \partial g$. A vertex $x \in V$ such that $\omega(e) = 0$ for all e adjacent to x is an *isolated vertex* of ω.

The probability $P_G(\omega)$ to see the system in the configuration $\omega \in \Omega_G$ is defined as

$$P_G(\omega) = \frac{1}{Z_G(p,q)} p^{|O(\omega)|}(1-p)^{|C(\omega)|} q^{k(\omega)}$$

where $p \in (0, 1)$, $q \in (0, \infty)$, and $k(\omega)$ is the number of connected open components of the configuration ω plus the number of isolated vertices; the normalization constant $Z_G(p, q)$, usually called the partition function of the system, is given by

$$Z_G^{\mathrm{RCM}}(p,q) = \sum_{\omega \in \Omega_G} p^{|O(\omega)|}(1-p)^{|C(\omega)|} q^{k(\omega)} \tag{3.1}$$

The "pressure" of the system is defined as the following function

$$\pi_G(p,q) = \frac{1}{|V|} \ln Z_G^{\mathrm{RCM}}(p,q)$$

In order to define the RCM on infinite graphs, we will need to introduce the concept of boundary condition. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ a connected and locally finite infinite graph and let $\Omega_{\mathbb{G}}$ be the set of all configurations in $\mathbb{G}$, i.e. the set of all functions ω such that $\omega : \mathbb{E} \to \{0, 1\}$. Let $V \subset \mathbb{V}$ a *finite* set and let $\mathbb{G}|_V$ be the restriction of $\mathbb{G}$ to V. Given now $\xi \in \Omega_{\mathbb{G}}$, let $\Omega^{\xi}_{\mathbb{G}|_V}$ the (finite) subset of $\Omega_{\mathbb{G}}$ of all configurations $\omega \in \Omega_{\mathbb{G}}$ such that $\omega(e) = \xi(e)$ for $e \notin \mathbb{E}|_V$. For $\omega \in \Omega^{\xi}_{\mathbb{G}|_V}$, let us also denote by ω_V the restriction of ω on $\mathbb{E}|_V$. Note that ω_V does not depend on ξ. We now denote $P^{\xi}_{\mathbb{G}|_V}$ the random cluster probability measure in $\Omega^{\xi}_{\mathbb{G}|_V}$ on the finite sub-graph $\mathbb{G}|_V$ of the infinite graph $\mathbb{G}$ with boundary conditions ξ as

$$P^{\xi}_{\mathbb{G}|_V}(\omega) = \frac{1}{Z^{\xi}_{\mathbb{G}|_V}(p,q)} p^{|O(\omega_V)|}(1-p)^{|C(\omega_V)|} q^{k^{\xi}_V(\omega)} \tag{3.2}$$

where $Z^{\xi}_{\mathbb{G}|_V}(p,q)$ is the partition function given by

$$Z^{\xi}_{\mathbb{G}|_V}(p,q) = \sum_{\omega \in \Omega^{\xi}_V} p^{|O(\omega_V)|}(1-p)^{|C(\omega_V)|} q^{k^{\xi}_V(\omega)} \tag{3.3}$$

and $k_V^\xi(\omega)$ is the number of *finite* connected open component (open clusters) of the configuration ω (which agrees with ξ outside V) which intersect V plus the number of isolated vertices in V. Note that $k_V^\xi(\omega)$ is the only term in (3.2) and (3.3) depending on boundary conditions ξ.
Two extremal boundary conditions play a central role, namely the *free boundary* condition, in which $\xi(e) = 0$ for all $e \in \mathbb{E}$ and the *wired boundary condition*, in which $\xi(e) = 1$ for all $e \in \mathbb{E}$. According to the definition above, for a fixed configuration ω with $\xi = 0$ outside V the number $k^0(\omega)$ is actually the number of open components in the finite sub graph $\mathbb{G}|_V$ plus the isolated vertices in V, while if $\xi = 1$ outside V, all open components in $\mathbb{G}|_V$ which touch the boundary have not to be counted computing the number $k^1(\omega)$, since they belong to the infinite open cluster. Thus $k^1(\omega)$ is actually the number of finite open connected component in ω which do not touch the boundary plus isolated vertices which do not belong to the boundary.

It is important to remark here that in the above definition of $k_V^\xi(\omega)$ we compute only the *finite* connected components because we are adopting the so called "infinity-wired boundary condition" convention, see e.g. definition 2.1 in [23] or section 2.3 in [21]. By this convention, all infinite open clusters eventually intersecting V are counted as one, i.e., as if all these clusters were connected at infinity (wired at infinity). In the literature one can also find the so-called "infinity-free boundary condition" convention, in which all open clusters, whether finite or infinite, are counted in the number $k(\omega)$. In this case all infinite clusters intersecting V are regarded as separate. This is e.g. the convention adopted in the survey [16] and in the book [17]. In the rest of the paper we will only consider the free ($\xi = 0$) and wired ($\xi = 1$) boundary conditions, for which the "infinity-free" convention and the "infinity-wired" convention are equivalent and we adopted the latter only because leads to simpler definitions.

Definition 3.1 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{B}$; let $\{V_N\}_{N\in\mathbb{N}}$ be a sequence of finite subsets of $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$ (not necessarily Følner); let ξ be a boundary condition. Then we define, if it exists and it is independent of $\{V_N\}_{N\in\mathbb{N}}$, the pressure of the random cluster model with parameters q and p and boundary condition ξ on $\mathbb{G}$ as*

$$\pi_{\mathbb{G}}^\xi(p,q) = \lim_{N\to\infty} \frac{1}{|V_N|} \ln Z_{\mathbb{G}|_{V_N}}^\xi(q) \tag{3.4}$$

In definition 3.1, instead of choosing a fixed boundary condition ξ, one can also think to allow a whole sequence ξ_N of boundary conditions, one for each $V_N \in \mathbb{V}$. However, as shown in [15] (see also [16, 17]), this adds no extra generality.

Remark 3.2 *With the further assumptions that $\mathbb{G}$ is amenable, quasi-transitive and the sequence $\{V_N\}_{N\in\mathbb{N}}$ is Følner, it is easy to prove that this limit, which is known to exist for all $q > 0$ and everywhere in the interval $p \in [0, 1]$ except possibly in a countable set of points (see [15, 23]), is independent of the boundary condition. As a matter of fact, let $\xi, \omega \in \Omega_{\mathbb{G}}$ and define ω_N^ξ by*

$$\omega_N^\xi(e) = \begin{cases} \omega(e) & \textit{if } e \in \mathbb{E}|_{V_N} \\ \\ \xi(e) & \textit{otherwise} \end{cases}$$

Then, for all ξ

$$k_{V_N}^1(\omega_N^1) \le k_{V_N}^\xi(\omega_N^\xi) \le k_{V_N}^0(\omega_N^0) \le k_{V_N}^1(\omega_N^1) + |\partial V_N|$$

whence

$$Z_{\mathbb{G}|_{V_N}}^1(p,q) \le Z_{\mathbb{G}|_{V_N}}^\xi(p,q) \le Z_{\mathbb{G}|_{V_N}}^0(p,q) \le Z_{\mathbb{G}|_{V_N}}^1(p,q) q^{|\partial V_N|}, \quad \text{if } q \ge 1$$

while for $q < 1$ we have simply to reverse all inequalities above. Now taking the logarithms, dividing by $|V_N|$, and using (2.7) one obtains the result.

Other important quantities to study are the so called connectivity functions. To introduce them we need some preliminary definitions.

Definition 3.3 *Let $\mathbb{G} \in \mathcal{G}$. An animal in $\mathbb{G}$ is a connected subgraph $g = (V_g, E_g)$ of $\mathbb{G}$ with vertex set V_g and edge set E_g such that $|V_g| < +\infty$ and $E_g \neq \emptyset$. We will denote by $A_{\mathbb{G}}$ the set of all animals in $\mathbb{G}$.*

Definition 3.4 *We say that two animals $g_1 = (V_{g_1}, E_{g_1})$ and $g_2 = (V_{g_2}, E_{g_2})$ in $\mathbb{G}$ are compatible and we write $g_1 \sim g_2$ if $V_{g_1} \cap V_{g_2} = \emptyset$ (hence consequently $E_{g_1} \cap E_{g_2} = \emptyset$). Otherwise we say that g_1 and g_2 are incompatible and write $g_1 \not\sim g_2$.*

We are now ready to give the definition of connectivity functions.

Definition 3.5 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{B}$ and let $X \subset \mathbb{V}$ finite. Let $\{V_N\}_{N\in\mathbb{N}}$ be a sequence of finite subsets of $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$ and $X \subset V_N$ for all $N \in \mathbb{N}$. Let ξ be a boundary condition. Then we define, if it exists and it is independent of $\{V_N\}_{N\in\mathbb{N}}$, the connectivity function of the set X of the random cluster model with parameters q and p and boundary condition ξ on $\mathbb{G}$ as*

$$\phi_{p,q,\xi}(X) = \lim_{N\to\infty} \sum_{\substack{\omega\in\Omega^{\xi}_{\mathbb{G}_N}:\ \exists g\in A_{\mathbb{G}}:\\ E_g\in O(\omega),\quad X\subseteq V_g}} P^{\xi}_{\mathbb{G}|_{V_N}}(\omega) \tag{3.5}$$

The finite connectivity function of the set X of the random cluster model with parameters q and p and boundary condition ξ is defined as

$$\phi^{\mathrm{f}}_{p,q,\xi}(X) = \lim_{N\to\infty} \sum_{\substack{\omega\in\Omega^{\xi}_{\mathbb{G}_N}:\ \exists g\in A_{\mathbb{G}}:\ E_g\in O(\omega)\\ X\subseteq V_g,\ V_g\cap\ \partial^{\mathrm{int}}_v V_N=\emptyset}} P^{\xi}_{\mathbb{G}|_{V_N}}(\omega) \tag{3.6}$$

In the r.h.s of (3.5) the sum runs over configurations ω containing an animal made by open edges whose vertex set contains X, while in the r.h.s of (3.6) the sum runs over configurations ω containing an animal made by open edges whose vertex set contains X and does not intersect the boundary of V_N.

Let us define the subcritical phase of a RCM on a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{B}$ at fixed q as the set of values of p in the interval $[0, 1]$ for which the probability to find an infinite open cluster in the system is zero. Conversely, the supercritical phase is the set of values of p in the interval $[0, 1]$ for which the probability to find an infinite open cluster in the system containing a fixed vertex is strictly greater than zero. We remark that $\phi^{\mathrm{f}}_{p,q,\xi}(X)$ coincides with $\phi_{p,q,\xi}(X)$ in the subcritical phase.
The connectivity function $\phi_{p,q,\xi}(X)$ is expected to decay exponentially to zero when $d^{\mathrm{tree}}_{\mathbb{G}}(X) \to \infty$ in the subcritical phase, while, of course, is not expected to decay to zero in the supercritical phase, where there is a non zero probability to find any set of vertices in the infinite cluster. The exponential decay of the connectivity function in the subcritical phase can be obtained for the RCM on $\mathbb{Z}^d$ in the regime $q \geq 1$ by comparison inequalities (see e.g. theorem 3.2 in [16]) and using the known results on Bernoulli bond percolation and/or Potts model. On the other hand, the finite connectivity function $\phi^{\mathrm{f}}_{p,q,\xi}(X)$ is expected to decay exponentially to zero when

$d_{\mathbb{G}}^{\rm tree}(X) \to \infty$ in the supercritical phase. Concerning again the RCM on $\mathbb{Z}^d$ in the regime $q \geq 1$, the exponential decay of finite connectivities (up to the slab percolation threshold in $d \geq 3$) follows from the renormalization group analysis developed in [32].
It is well known (see e.g. theorem 3.6 in [16]) that, for $q \geq 1$ we have, by FKG inequalities, that

$$\phi_{p,q,0}(X) \leq \phi_{p,q,\xi}(X) \leq \phi_{p,q,1}(X) \tag{3.7}$$

$$\phi^{\rm f}_{p,q,0}(X) \leq \phi^{\rm f}_{p,q,\xi}(X) \leq \phi^{\rm f}_{p,q,1}(X) \tag{3.8}$$

for any boundary condition ξ. Hence if one is able to prove that

$$\phi_{p,q,1}(X) = \phi_{p,q,0}(X),$$

and/or

$$\phi^{\rm f}_{p,q,1}(X) = \phi^{\rm f}_{p,q,0}(X),$$

then automatically $\phi_{p,q,1}(X) = \phi_{p,q,\xi}(X) = \phi_{p,q,0}(X)$ and/or $\phi^{\rm f}_{p,q,1}(X) = \phi^{\rm f}_{p,q,\xi}(X) = \phi^{\rm f}_{p,q,0}(X)$ for any fixed the boundary condition ξ, *as far as* $q \geq 1$. We stress that when $q < 1$ we cannot get to the same conclusion, since (3.7) and (3.8) are false when $q < 1$.

As it will be shown below we are able to prove using cluster expansion techniques for all $q > 0$ that $\phi_{p,q,1}(X) = \phi_{p,q,0}(X)$ for p sufficiently small and that $\phi^{\rm f}_{p,q,1}(X) = \phi^{\rm f}_{p,q,0}(X)$ for p sufficiently near 1. It is unclear for us if it is possible to generalize our expansions in order to include all boundary conditions in the whole regime $q > 0$. For these reasons we preferred to treat only the simplest and most popular case $\xi = 0, 1$.
Note finally that, given a vertex $x_0 \in \mathbb{V}$, the percolation probability $\theta^{\xi}_{p,q}(x_0 \leftrightarrow \infty)$, i.e. the probability that there is an infinite open cluster passing through x_0 is defined in term of the 1-point finite connectivity function as

$$\theta^{\xi}_{p,q}(x_0 \leftrightarrow \infty) = 1 - \phi^{\rm f}_{p,q,\xi}(x_0) \tag{3.9}$$

The critical percolation probability $p^{\xi}_c(q)$ at a fixed value of q for the graph $\mathbb{G}$ is the value of p defined by

$$p^{\xi}_c(q) = \sup_{\substack{p\in[0,1]\\ x_0\in\mathbb{V}}} \{p : \theta^{\xi}_{p,q}(x_0 \leftrightarrow \infty) = 0\} \tag{3.10}$$

We recall that for the RCM on $\mathbb{Z}^d$ and $q \geq 1$ theorem 4.2 of [1] states that $p^{\xi}_c(q)$ is independent of boundary conditions and strictly smaller than 1, while results of [34] imply for the RCM on $\mathbb{Z}^d$ with $q < 1$ that $p^{0,1}_c(q) < 1$. We also recall that for the particular case of $\mathbb{Z}^2$, duality arguments lead to the conjecture that $p_c(q) = \sqrt{q}/(1+\sqrt{q})$. This conjecture has proven to be true for $q = 1$ [28], $q = 2$ [31] and for q sufficiently large [24].

4 The subcritical phase

4.1 Results in the subcritical phase

We begin this section stating our two main theorems about subcritical phase. The first theorem concerns the connectivity functions. The second concerns the pressure. The rest of the section will be devoted to the proof of these two theorems.

Theorem 4.1 *Let $\mathbb{G} \in \mathcal{B}$ with maximum degree Δ. For any $q > 0$, let $\{V_N\}_{N\in\mathbb{N}}$ be any sequence in $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$ (it does not need to be Følner), and let p be so small that $(3+2\sqrt{2})\varepsilon_p \leq 1$ where*

$$\varepsilon_p = \max\left\{\frac{e\Delta}{q}\left|\frac{\ln(1-p)}{(1-p)^\Delta}\right|,\ e\Delta\left|\frac{\ln(1-p)}{(1-p)^\Delta}\right|\right\} \tag{4.1}$$

Then the infinite volume connectivity functions $\phi_{p,q,\xi}(X)$ with $\xi = 0, 1$ of the RCM on $\mathbb{G}$ defined in the limit (3.5) exist, are both equal to a function $\phi_{p,q}(X)$ which can be written explicitly in terms of an absolutely convergent series which is analytic as a function of p, and does not depend on the sequence V_N.
Moreover $|\phi_{p,q}(X)|$ admits the upper bound

$$|\phi_{p,q}(X)| \leq \frac{(7+5\sqrt{2})}{(2\sqrt{2}+3)}\left[\left(1+\frac{1}{\sqrt{2}}\right)\varepsilon_p\right]^{d_{\mathbb{G}}^{\rm tree}(X)-1} \tag{4.2}$$

where $d_{\mathbb{G}}^{\rm tree}(X)$ is the tree distance of X in $\mathbb{G}$ accordingly to definition 2.1.

Theorem 4.2 *Let $\mathbb{G} \in \mathcal{B} \cap \mathcal{A} \cap \mathcal{Q}^v$ with maximum degree Δ. Let $q > 0$ be fixed, let $\{V_N\}_{N\in\mathbb{N}}$ any Følner sequence in $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$, and let p so small that $2e^2\varepsilon_p^* < 1$ where*

$$\varepsilon_p^* = \frac{e\Delta}{q}\left|\frac{\ln(1-p)}{(1-p)^\Delta}\right| \tag{4.3}$$

Then the pressure of Random Cluster Model on $\mathbb{G}$, defined in (3.4) exists and can be written explicitly in term of an absolutely convergent series which is analytic as a function of p, and does not depend on V_N and on ξ.

Note that the first theorem, concerning connectivity functions, holds for a larger class of graphs, but in a smaller region of parameters, while theorem 4.2 concerning the pressure is valid for a smaller class of graphs, which however includes all regular lattices, but in a larger region of the parameters p and q.
Once again we recall that the existence of these limits and independency of boundary conditions is well known for the RCM on $\mathbb{Z}^d$ for $q \geq 1$ in the whole interval $p \in [0, 1]$, except in a subset at most countably infinite (conjectured to be a singleton or empty), see e.g. theorem 3.6 in [16].

4.2 Proof of theorem 4.1. Polymer expansion for the connectivity functions

In this section we will assume that $\mathbb{G} \in \mathcal{B}$. Let us take sequence $\{V_N\}_{N\in\mathbb{N}}$ in $\mathbb{V}$ tending monotonically to $\mathbb{V}$. We will use the shorter notations $\mathbb{G}_N = \mathbb{G}|_{V_N}$ and $\mathbb{E}_N = \mathbb{E}|_{V_N}$, $k^\xi_{V_N} = k^\xi_N$ and also $\omega_{\mathbb{E}_N} = \omega_N$.
Fix a $X \subset V_N - \partial_v^{\rm int} V_N$ (i.e., X does not touch the boundary). The finite volume free and wired connectivity function can be rewritten as

$$\phi^N_{p,q,\xi=0,1}(X) = \frac{1}{\tilde{Z}^\xi_{\mathbb{G}|_N}(p,q)} \sum_{\substack{\omega\in\Omega^\xi_{\mathbb{G}_N}:\ \exists g\in A_{\mathbb{G}}: \\ E_g\subset O(\omega),\quad X\subset V_g}} \lambda^{|O(\omega_N)|} q^{k^\xi_N(\omega)} \tag{4.4}$$

where

$$\tilde{Z}^\xi_{\mathbb{G}_N}(p,q) = \sum_{\omega\in\Omega^\xi_{\mathbb{G}_N}} \lambda^{|O(\omega_N)|} q^{k^\xi_N(\omega)} = (1-p)^{|\mathbb{E}_N|} Z^\xi_{\mathbb{G}_N}(p,q) \tag{4.5}$$

and

$$\lambda = \frac{p}{1-p} \tag{4.6}$$

We recall that $k_N^0(\omega)$ is the number of open components of ω_N plus isolated vertices, while $k_N^1(\omega)$ is the number of open connected component in ω_N which do not intersect the boundary plus isolated vertices which does belong to the boundary $\partial_v^{\rm int} V_N$.

A configuration $\omega \in \Omega^{\xi}_{\mathbb{G}_N}$ is completely specified by the set of open edges $O(\omega_N)$ in $\mathbb{E}_N$. Let now $\{E_1, \ldots, E_n\}$ be the connected components of $O(\omega_N)$. To each E_i we can associate an animal $g_i \in \mathcal{A}_{\mathbb{G}_N}$ such that $V_{g_i} = \mathbb{V}|_{E_i}$, $E_{g_i} = E_i$. Then to each $\omega \in \Omega^{\xi}_{\mathbb{G}_N}$ can be associated a (unordered) set of animals $\{g_1, \ldots, g_n\}_{\omega_N} \subset \mathcal{A}_{\mathbb{G}_N}$ such that $\cup_{i=1}^n E_{g_i} = O(\omega_N)$ and for all $i, j \in \mathrm{I}_n$, $g_i \sim g_j$. Observe that this one to one correspondence $\omega_N \leftrightarrow \{g_1, \ldots, g_n\}$ yields

$$|O(\omega_N)| = \sum_{i=1}^n |E_{g_i}| \tag{4.7}$$

$$\sum_{\omega \in \Omega^{\xi}_{\mathbb{G}_N}} (\cdot) = \sum_{n \geq 0} \sum_{\substack{\{g_1, \ldots, g_n\} \subset \mathcal{A}_{\mathbb{G}_N} \\ g_i \sim g_j}} (\cdot) \tag{4.8}$$

$$\sum_{\substack{\omega \in \Omega^{\xi}_{\mathbb{G}_N} :\ \exists g \in A_{\mathbb{G}}: \\ E_g \subset O(\omega),\quad X \subset V_g}} (\cdot) = \sum_{n \geq 1} \sum_{\substack{\{g_1, \ldots, g_n\} \subset \mathcal{A}_{\mathbb{G}_N} \\ g_i \sim g_j,\ X \subset V_{g_1}}} (\cdot) \tag{4.9}$$

where for $n = 0$ the unordered n-uple $\{g_1, \ldots, g_n\}$ is the empty set.
We will now rewrite the partition function (4.5) and the connectivity function (4.4) in terms of the animals introduced above. We start by considering the case $\xi = 0$. Let us denote by $V^{\rm iso}_{\omega_N}$ the subset of V_N formed by the isolated vertices in the configuration ω_N, and let $\{g_1, \ldots, g_n\}_{\omega_N}$ be the animals uniquely associated to $O(\omega_N)$. Then, by definition,

$$k_N^0(\omega) = n + |V^{\rm iso}_{\omega_N}|$$

and since

$$|V^{\rm iso}_{\omega_N}| = |V_N| - \sum_{i=1}^n |V_{g_i}|$$

we obtain

$$k_N^0(\omega) = |V_N| - \sum_{i=1}^n \Big[|V_{g_i}| - 1) \Big] \tag{4.10}$$

Using now (4.7), (4.8), (4.9) and (4.10), the partition function $\tilde{Z}^0_{\mathbb{G}_N}(p, q)$ defined in (4.5) can be rewritten as

$$\tilde{Z}^0_{\mathbb{G}_N}(p, q) = q^{|V_N|} \Xi^0_{\mathbb{G}_N}(p, q) \tag{4.11}$$

where

$$\Xi^0_{\mathbb{G}_N}(p, q) = 1 + \sum_{n \geq 1} \sum_{\substack{\{g_1, \ldots, g_n\} \subset \mathcal{A}_{\mathbb{G}_N} \\ g_i \sim g_j}} \prod_{i=1}^n \frac{1}{q^{|V_{g_i}| - 1}} \lambda^{|E_{g_i}|} \tag{4.12}$$

and

$$\phi^N_{p,q,\xi=0}(X) = \frac{1}{\Xi^0_{\mathbb{G}_N}(p,q)} \sum_{n\geq 1} \sum_{\substack{\{g_1,\dots,g_n\}\subset\mathcal{A}_{\mathbb{G}_N}\\ g_i\sim g_j,\ X\subset V_{g_1}}} \prod_{i=1}^{n} \frac{1}{q^{|V_{g_i}|-1}} \lambda^{|E_{g_i}|}$$

The case $\xi = 1$ is slightly more involved. We first find an expression of $k^1_N(\omega)$ in terms of the animals $\{g_1, \dots g_n\}$. The set $I_n = \{1, 2, \dots, n\}$ is naturally partitioned in the disjoint union of two sets $\mathrm{I}^{\rm int}_n$ and I^{∂}_n defined as

$$\mathrm{I}^{\rm int}_n = \{i \in \mathrm{I}_n : V_{g_i} \cap \partial^{\rm int}_v V_N = \emptyset\}$$

$$\mathrm{I}^{\partial}_n = \{i \in \mathrm{I}_n : V_{g_i} \cap \partial^{\rm int}_v V_N \neq \emptyset\}$$

With these notations, denoting shortly $V_N - \partial^{\rm int}_v V_N = V^{\rm int}_N$ and, for $i \in \mathrm{I}^{\partial}_n$, $V^{\rm int}_{g_i} = V_{g_i} - \partial^{\rm int}_v V_N$, we have

$$k^1_N(\omega) = |V^{\rm int}_N| - \sum_{i\in\mathrm{I}^{\rm int}_n} (|V_{g_i}| - 1) - \sum_{i\in\mathrm{I}^{\partial}_n} |V^{\rm int}_{g_i}| \tag{4.13}$$

Hence in the case $\xi = 1$ we get

$$\tilde{Z}^1_{\mathbb{G}_N}(p,q) = q^{|V^{\rm int}_N|}\Xi^1_{\mathbb{G}_N}(p,q)$$

where

$$\Xi^1_{\mathbb{G}_N}(p,q) = 1 + \sum_{n\geq 1} \sum_{\substack{\{g_1,\dots,g_n\}\subset\mathcal{A}_{\mathbb{G}_N}\\ g_i\sim g_j}} \prod_{i\in\mathrm{I}^{\rm int}_n} \frac{1}{q^{|V_{g_i}|-1}} \lambda^{|E_{g_i}|} \prod_{i\in\mathrm{I}^{\partial}_n} \frac{1}{q^{|V^{\rm int}_{g_i}|}} \lambda^{|E_{g_i}|}$$

and

$$\phi^N_{p,q,\xi=1}(X) = \frac{1}{\Xi^1_{\mathbb{G}_N}(p,q)} \sum_{n\geq 1} \sum_{\substack{\{g_1,\dots,g_n\}\subset\mathcal{A}_{\mathbb{G}_N}\\ g_i\sim g_j,\ X\subset V_{g_1}}} \prod_{i\in\mathrm{I}^{\rm int}_n} \frac{1}{q^{|V_{g_i}|-1}} \lambda^{|E_{g_i}|} \prod_{i\in\mathrm{I}^{\partial}_n} \frac{1}{q^{|V^{\rm int}_{g_i}|}} \lambda^{|E_{g_i}|}$$

We now rewrite $\phi^N_{p,q,\xi=1}(X)$ in term of a polymer expansion in which polymers are finite subsets of $\mathbb{V}$ with cardinality greater than 1 which are said to be incompatible in the usual polymer expansion terminology if they overlap.

Let us now define, for each pair $\{x, y\} \subset \mathbb{V}$,

$$V_{xy} = \begin{cases} 0 & \text{if } \{x,y\} \notin \mathbb{E} \\ \\ \ln(1+\lambda) & \text{if } \{x,y\} \in \mathbb{E} \end{cases}$$

Let us also define, for any subset $R \subset \mathbb{V}$ such that $2 \leq |R| < +\infty$, the activity

$$\rho(R) = q^{-(|R|-1)} \sum_{\substack{E'\subset\mathrm{P}_2(R)\\ (R,E')\in\mathcal{G}_R}} \prod_{\{x,y\}\in E'} (e^{V_{xy}} - 1) \tag{4.14}$$

where $\mathcal{G}_R$ is the set of connected graphs with vertex set R. For $R \subset V_N$ we also define a ξ-dependent set activity as

$$\rho^\xi(R)=\begin{cases}\rho(R) & \text{if } \xi=0\\ \rho(R) & \text{if } \xi=1 \text{ and } R\cap\partial_v^{\rm int}V_N=\emptyset\\ q^{-|R\cap V_N^{\rm int}|}\sum\limits_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{G}_R}}\prod\limits_{\{x,y\}\in E'}(e^{V_{xy}}-1) & \text{if } \xi=1 \text{ and } R\cap\partial_v^{\rm int}V_N\neq\emptyset\end{cases} \tag{4.15}$$

Note that $\rho^0(R)$ is the restriction of $\rho(R)$ for $R\subset\mathbb{E}_N$ and when $q<1$ we have, for all $R\in P_{\ge 2}(V_N)$, that

$$|\rho^\xi(R)|\le|\rho(R)| \qquad \text{whenever } \ q<1 \tag{4.16}$$

Note also that

$$\rho^\xi(R)=0 \qquad \text{whenever } R \text{ is not connected in } \mathbb{G}$$

We are thus ready to define our polymer space.

Definition 4.3 *We define the set of (subcritical) polymers as the set*

$$\mathcal{P}=\{R\subset\mathbb{V}:\ 2\le|R|<+\infty,\quad R \textit{ is connected in } \mathbb{G}\}$$

We will say that two polymers $R_i,R_j\in\mathcal{P}$ are compatible, and we write $R_i\sim R_j$, if $R_i\cap R_j=\emptyset$; viceversa, R_i and R_j are incompatible, and we write $R_i\not\sim R_j$, if $R_i\cap R_j\neq\emptyset$. For $V_N\subset\mathbb{V}$ finite we define

$$\mathcal{P}_N=\{R\subset\mathbb{V}_N: |R|\ge 2,\quad R \textit{ is connected in } \mathbb{G}\}$$

Then for $\xi=0,1$ we can write

$$\phi^N_{p,q,\xi}(X)=\frac{1}{\Xi^\xi_{\mathbb{G}|_N}(p,q)}\sum_{n\ge1}\frac{1}{n!}\sum_{\substack{(R_1,\dots,R_n)\in\mathcal{P}_N^n\\ R_i\sim R_j,\ \exists i\in \mathrm{I}_n:\ R_i\supset X}}\rho^\xi(R_1)\cdots\rho^\xi(R_n) \tag{4.17}$$

where $\mathrm{I}_n=\{1,2,\cdots,n\}$ and $\mathcal{P}^n$ is the n-times cartesian product of $\mathcal{P}$, i.e. elements of $\mathcal{P}^n_N$ are ordered n-ples of elements of $\mathcal{P}_N$. The partition function $\Xi^\xi_{\mathbb{G}|_N}(p,q)$ can be rewritten as

$$\Xi^\xi_{\mathbb{G}_N}(p,q)=\left[1+\sum_{n\ge1}\frac{1}{n!}\sum_{\substack{(R_1,\dots,R_n)\in\mathcal{P}_N^n\\ R_i\sim R_j}}\rho^\xi(R_1)\cdots\rho^\xi(R_n)\right] \tag{4.18}$$

The factor 1 in r.h.s. is the contribution of the configuration in which all edges in $\mathbb{G}_N$ are closed. Observe that the partition function is rewritten as a genuine Gruber and Kunz hard core polymer gas partition function in which the polymers are finite subsets R of V_N with cardinality greater than one and with activity $\rho^\xi(R)$.
It is now easy to rewrite this ratio (between two finite sums) as an infinite series. Define, for $R\in\mathcal{P}$

$$\Pi^N_{p,q,\xi}(R)=\frac{\partial}{\partial\rho^\xi(R)}\ln\left[\Xi^\xi_{\mathbb{G}_N}(p,q)\right]$$

Then, by construction

$$\phi^N_{p,q,\xi}(X) = \sum_{\substack{R\in\mathcal{P}_N\\ X\subset R}} \rho^\xi(R)\Pi^N_{p,q,\xi}(R) \tag{4.19}$$

Now, by standard cluster expansion it is well known that

$$\ln \Xi^\xi_{\mathbb{G}_N}(p,q) = \sum_{n\geq 1}\frac{1}{n!}\sum_{(R_1,\ldots,R_n)\in\mathcal{P}^n_N} \rho^\xi(R_1)\cdots\rho^\xi(R_n)\Phi^T(R_1,\ldots,R_n) \tag{4.20}$$

where the Ursell coefficients $\Phi^T(R_1,\ldots,R_n)$ are given by

$$\Phi^T(R_1,\ldots,R_n) = \begin{cases} \sum\limits_{\substack{E\subset E(R_1,\ldots,R_n)\\ (\mathrm{I}_n,E)\in\mathcal{G}_n}} (-1)^{|E|} & \text{if } n\geq 2\\ \\ 1 & \text{if } n=1. \end{cases} \tag{4.21}$$

where $E(R_1,\ldots,R_n) = \{\{i,j\}\subset \mathrm{I}_n : R_i \not\sim R_j\}$ and $\mathcal{G}_n$ denotes the set of all connected graphs with vertex set I_n. So

$$\Pi^N_{p,q,\xi}(R) = \sum_{n\geq 0}\frac{1}{n!}\sum_{(R_1,\ldots,R_n)\in\mathcal{P}^n_N} \rho^\xi(R_1)\cdots\rho^\xi(R_n)\Phi^T(R,R_1,\ldots,R_n) \tag{4.22}$$

We also define functions on the whole $\mathbb{G}$ (hence not depending on boundary conditions) as follows

$$\Pi_{p,q}(R) = \sum_{n\geq 0}\frac{1}{n!}\sum_{(R_1,\ldots,R_n)\in\mathcal{P}^n} \rho(R_1)\cdots\rho(R_n)\Phi^T(R,R_1,\ldots,R_n) \tag{4.23}$$

$$\phi_{p,q}(X) = \sum_{\substack{R\in\mathcal{P}\\ X\subset R}} \rho(R)\Pi_{p,q,}(R) \tag{4.24}$$

We can now use the methods of the abstract polymer gas, see [29, 10] to determine the convergence radius for the series (4.22) and (4.23) and their bounds. We will see that this formal series are indeed an absolutely convergent expansions for the infinite volume connectivity functions for p sufficiently small.

4.3 Proof of theorem 4.1. Convergence of the connectivity functions

First we prove an exponential bound on the activity $\rho(R)$, which is an essential ingredient for the convergence of the cluster expansion.

Lemma 4.4 *Let $\mathbb{G}\in\mathcal{B}$ with maximum degree Δ. Then, for any $n\geq 2$ and $\xi = 0,1$*

$$\sup_{x\in\mathbb{V}} \sum_{\substack{R\in\mathcal{P}:\\ x\in R,\ |R|=n}} |\rho(R)| \leq (\varepsilon^*_p)^{n-1} \leq \varepsilon_p^{n-1} \tag{4.25}$$

and,

$$\sup_{x\in V_N} \sum_{\substack{R\in\mathcal{P}_N\\ x\in R,\ |R|=n}} |\rho^\xi(R)| \leq \varepsilon_p^{n-1} \tag{4.26}$$

where ε_p and ε^*_p are defined in (4.1) and (4.3) respectively.

Proof. Observe that, for $R \in P_{\geq 2}(\mathbb{V})$ by definition of (4.14)

$$\sup_{x\in\mathbb{V}} \sum_{\substack{R\in P_{\geq 2}(\mathbb{V}):\ x\in R\\ |R|=n}} |\rho(R)| \leq |q|^{-(n-1)} \sup_{x\in\mathbb{V}} \sum_{\substack{R\in P_n(\mathbb{V})\\ x\in R}} \left| \sum_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{G}_R}} \prod_{\{x,y\}\in E'} \left[e^{V_{xy}}-1\right]\right| \tag{4.27}$$

while, for $\rho^{\xi}(R)$ we have in the worst case (i.e. for $R \subset \partial_v V_N^{\mathrm{int}}$)

$$\sup_{x\in V_N} \sum_{\substack{R\in P_{\geq 2}(V_N)\\ x\in R,\ |R|=n}} |\rho^{\xi}(R)| \leq \sup_{x\in\mathbb{V}} \sum_{\substack{R\in P_n(\mathbb{V}):\\ x\in R}} \left| \sum_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{G}_R}} \prod_{\{x,y\}\in E'} [e^{V_{xy}}-1]\right| \tag{4.28}$$

Then all we have to show to prove the lemma is that

$$\sup_{x\in\mathbb{V}} \sum_{\substack{R\in P_n(\mathbb{V}):\\ x\in R}} \left| \sum_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{G}_R}} \prod_{\{x,y\}\in E'} [e^{V_{xy}}-1]\right| \leq (e|f_\Delta(p)|)^{n-1}$$

Using thus the Battle-Brydges-Federbush inequality (see e.g. [9]), recalling that $\mathbb{E}|_R = \{\{x,y\} \in \mathbb{E} : x \in R, y \in R\}$, and observing that $\sum_{\{x,y\}\in R} V_{xy} \leq \frac{1}{2}\Delta|R| \leq \Delta(|R|-1)$ for all R such that $|R| \geq 2$, we get

$$\left| \sum_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{G}_R}} \prod_{\{x,y\}\in E'} [e^{V_{xy}}-1]\right| \leq [(1+\lambda)^{\Delta}\ln(1+\lambda)]^{|R|-1} \sum_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{T}_R}} \prod_{\{x,y\}\in E'} \delta_{|x-y|1}$$

where $\mathcal{T}_R$ is the set of all connected *tree graphs* with vertex set R and $\delta_{|x-y|1} = 1$ if $|x-y| = 1$ and $\delta_{|x-y|1} = 0$ otherwise. It is now easy to check that

$$\sum_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{T}_R}} \prod_{\{x,y\}\in E'} \delta_{|x-y|1} \leq \sup_{x\in\mathbb{V}} \sum_{\substack{R\in P_n(\mathbb{V}):\\ x\in R}} \sum_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{T}_R}} \prod_{\{x,y\}\in E'} \delta_{|x-y|1} \leq$$

$$\leq \frac{1}{(n-1)!} \sum_{\substack{E'\subset \mathrm{P}_2(\mathrm{I}_n)\\ (\mathrm{I}_n,E')\in\mathcal{T}_n}} \left[\sup_{x\in\mathbb{V}} \sum_{\substack{x_1=x,\,(x_2,\ldots,x_n)\in\mathbb{V}^{n-1}\\ x_i\neq x_j\ \forall\{i,j\}\in\mathrm{I}_n}} \prod_{\{i,j\}\in E'} \delta_{|x_i-x_j|1}\right]$$

Now observe that, for any $E' \subset \mathrm{P}_2(\mathrm{I}_n)$ such that (I_n, E') is a tree, it holds

$$\sup_{x\in\mathbb{V}} \sum_{\substack{x_1=x,\,(x_2,\ldots,x_n)\in\mathbb{V}^{n-1}\\ x_i\neq x_j\ \forall\{i,j\}\in\mathrm{I}_n}} \prod_{\{i,j\}\in E'} \delta_{|x_i-x_j|1} \leq \Delta^{n-1}$$

Moreover,using Cayley formula, $|\{E' \subset \mathrm{P}_2(\mathrm{I}_n) :\ (R,E') \in \mathcal{T}_n\}| = n^{n-2}$, and the estimate $n^{n-2}/(n-1)! \leq e^{n-1}$, we can conclude that

$$\sup_{x\in\mathbb{V}} \sum_{\substack{R\in P_n(\mathbb{V}):\\ x\in R}} \left| \sum_{\substack{E'\subset \mathrm{P}_2(R)\\ (R,E')\in\mathcal{G}_R}} \prod_{\{x,y\}\in E'} [e^{V_{xy}}-1]\right| \leq \left[e\Delta(1+\lambda)^{\Delta}\ln(1+\lambda)\right]^{(n-1)}$$

$\square$

Using this result one can the prove the following lemma

Lemma 4.5 *For any $q > 0$, the function $\phi_{p,q}(X)$ defined in (4.24) is analytic as a function of p whenever $(3+2\sqrt{2})\varepsilon_p \le 1$ where ε_p is the number in (4.1) and satisfies the bound (4.2), uniformly in V_N and $\xi = 0, 1$. Moreover the function $\phi^N_{p,q,\xi}(X)$ defined in (4.4) is also analytic as a function of p whenever $(3+2\sqrt{2})\varepsilon \le 1$ and $|\phi^N_{p,q,\xi}(X)|$ is bounded above by the r.h.s. of (4.2).*

Proof. Using the condition (3.16) of [10], valid for polyemers whose incompatibility relation is the overlapping, we have that the series (4.23) converges if

$$\sup_{x\in\mathbb{V}} \sum_{\substack{R\in\mathcal{P}\\ x\in R}} |\rho(R)|e^{a|R|} \le e^a - 1 \tag{4.29}$$

Using lemma 4.4 we have that

$$\sup_{x\in\mathbb{V}} \sum_{\substack{R\in\mathcal{P}\\ x\in R}} |\rho(R)|e^{a|R|} \;\le\; \sum_{n\ge 2} e^{a|n|} \sup_{x\in\mathbb{V}} \sum_{\substack{R\in\mathcal{P}\\ x\in R:|R|=n}} |\rho(R)| \;\le\; \sum_{n\ge 2} e^{a|n|}\varepsilon^{n-1}$$

So condition (4.29) is optimal for $a = \ln(1+\frac{1}{\sqrt{2}})$ and gives

$$\varepsilon \le \frac{1}{3+2\sqrt{2}} \tag{4.30}$$

This for ε satisfying (4.30) the series (4.22) and (4.23) are convergent and, by theorem 1 of [10] (see there formula (3.17)) we have the bound

$$\Pi_{p,q,}(R) \le e^{a|R|} \le \left(1+\frac{1}{\sqrt{2}}\right)^{|R|}$$

So, recalling (4.19) and (4.24) and observing that $\min\{|R| : R\in\mathcal{P},\ X\subset R\} = d^{\rm tree}_{\mathbb{G}}(X)$, we get

$$|\phi_{p,q,}(X)| = \sum_{\substack{R\in\mathcal{P}\\ X\subset R}} |\rho(R)| \left(1+\frac{1}{\sqrt{2}}\right)^{|R|} = \sum_{n\ge d^{\rm tree}_{\mathbb{G}}(X)} \varepsilon_p^{n-1}\left(1+\frac{1}{\sqrt{2}}\right)^{n} \le$$

$$\left(1+\frac{1}{\sqrt{2}}\right) \sum_{n\ge d^{\rm tree}_{\mathbb{G}}(X)-1} \left[\varepsilon_p\left(1+\frac{1}{\sqrt{2}}\right)\right]^n \le \frac{(7+5\sqrt{2})}{(2\sqrt{2}+3)}\left[\frac{\varepsilon_p(\sqrt{2}+1)}{\sqrt{2}}\right]^{d^{\rm tree}_{\mathbb{G}}(X)-1}$$

The proof that $\phi^N_{p,q,\xi}(X)$ is also analytic and $|\phi^N_{p,q,\xi}(X)|$ admits the same upper bound (4.2) is completely analogous just observing that, by (4.26) and (4.1), $\sup_{x\in\mathbb{V}}\sum_{R\ni x:|R|=n}|\rho^\xi(R)|$ admits the same bound of $\sup_{x\in\mathbb{V}}\sum_{R\ni x:|R|=n}|\rho(R)|$. $\square$
Finally we prove the following result which ends the proof of theorem 4.1.

Lemma 4.6 *Let $\mathbb{G} = (\mathbb{V},\mathbb{E})$ be a bounded degree graph and let $\{V_N\}$ be any sequence in $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$. Then for any fixed $q > 0$, $\xi = 0, 1$ and p such that $(3+2\sqrt{2})\varepsilon_p \le 1$*

$$\lim_{N\to\infty} \phi^N_{p,q,\xi}(X) = \phi_{p,q}(X)$$

where $\phi_{p,q}(X)$ is the function defined in (4.24).

To prove this theorem we will first need to prove a simple graph theory lemma stated as follows.

Lemma 4.7 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be bounded degree, let $V_N \nearrow \mathbb{V}$ be a sequence of finite subsets tending monotonically to $\mathbb{V}$, and let x a vertex of $\mathbb{G}$ such that $x \in V_N$ for all N, then*

$$\lim_{N\to\infty} d(x, \partial_v^{\rm int} V_N) = +\infty$$

Proof. Suppose that it is possible to find $x_0 \in \bigcap_N V_N$ such that $d(x_0, \partial_v^{\rm int} V_N) < R$ for some real constant R. Then one can construct an infinite sequence $\{x_N\}_{N\in\mathbb{N}}$ of *distinct* vertices such that $x_N \in V_N$ but $x_N \notin V_M$ for all $M < N$ and $d(x_0, x_N) \leq R$ for all x_N. So this means that all x_N are in the ball of radius R and center x_0. But since $\mathbb{G}$ is bounded degree this ball is finite and we have a contradiction. $\Box$

We are now ready to prove the lemma 4.6.

Proof of lemma 4.6. Let us consider the case $\xi = 1$, which is the less trivial case.

$$|\phi_{p,q}(X) - \phi^N_{p,q,\xi=1}(X)| =$$

$$= \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{(R_1,\dots,R_n)\in\mathcal{P}^n \\ X\subset R_1, \exists j:\; R_j \not\subset V_N}} \rho(R_1)\cdots\rho(R_n)\Phi^T(R_1,\dots,R_n) +$$

$$+ \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{(R_1,\dots,R_n)\in\mathcal{P}^n_N \\ X\subset R_1, \exists j:\; R_j\cap\partial_v^{\rm int} V_N \neq\emptyset}} |\rho(R_1)\cdots\rho(R_n) - \rho^1(R_1)\cdots\rho^1(R_n)|\Phi^T(R_1,\dots,R_n)$$

Now, the first term of the r.h.s. of this inequality is, for $(3+2\sqrt{2})\varepsilon_p \leq 1$, clearly at least of the order $([1+1/\sqrt{2}]\varepsilon_p)^{d_{\mathbb{G}}(X,\partial_v^{\rm int} V_N)}$, with since one among the $R_1,\dots,R_n$ has to contain X and another has to intersect $\mathbb{V} - V_N$. Recall that the sets $R_1,\dots,R_n$ are pairwise intersecting due to the presence of the factor $\Phi^T(\mathbf{R}_n)$.

The second term can be treated similarly, due to the bounds (4.25) and (4.26), and again one shows that it is of the order $([1+1/\sqrt{2}]\varepsilon_p)^{d_{\mathbb{G}}(X,\partial_v^{\rm int} V_N)}$. Now as $N\to\infty$ we have clearly that $d_{\mathbb{G}}(X,\partial_v^{\rm int} V_N)\to\infty$ due to lemma 4.7. The proof of the case $\xi = 0$ is the same, since just the first term in the inequality above is present. $\Box$

4.4 Proof of theorem 4.2

To prove theorem 4.2, we recall that the pressure of the random cluster model is given by (3.4). As it has been shown in the remark 3.2, if the pressure exists, it is independent of boundary conditions. Hence we can work here with free boundary conditions $\xi = 0$ which are easier for small p.

Now by (4.5) and (4.11)

$$\frac{1}{|V_N|}\ln Z^0_{\mathbb{G}|_{V_N}}(q) = \frac{1}{|V_N|}\ln \Xi^0_{\mathbb{G}|_{V_N}}(q) - \frac{|\mathbb{E}_N|}{V_N}\ln(1-p) + \ln q$$

where we recall that $\Xi^\xi_{\mathbb{G}_N}(p,q)$ is given explicitly by equation (4.18).

We have

Proposition 4.8 *Let $\mathbb{G}$ amenable and quasi-transitive with vertex orbits $O_1, \ldots, O_k$, let Δ_i be the degree of the vertices in the orbit O_i (for $i = 1, \ldots, k$), and let $\{V_N\}_{N\in\mathbb{N}}$ be a Følner sequence such that $V_N \nearrow \mathbb{V}$. Then, there exists a non-zero finite limit*

$$\lim_{N\to\infty} \frac{|\mathbb{E}_N|}{|V_N|} \tag{4.31}$$

independent of the choice of the Følner sequence $\{V_N\}_{N\in\mathbb{N}}$.

Proof. By lemma 6 of [35] the limit

$$\lim_{N\to\infty} \frac{|O_i \cap V_N|}{|V_N|} = \alpha_i$$

exists and it is independent of the choice of the sequence $\{V_N\}_{N\in\mathbb{N}}$. Hence, considering that each vertex in an orbit O_i has Δ_i edges and each of these edges counts $1/2$ since it is shared with another vertex, one obtains immediately that

$$\lim_{N\to\infty} \frac{|\mathbb{E}_N|}{|V_N|} = \frac{1}{2}(\alpha_1\Delta_1 + \ldots + \alpha_k\Delta_k) \tag{4.32}$$

□

By this proposition we have that

$$\pi_{\mathbb{G}}(p,q) = \lim_{N\to\infty} \frac{1}{|V_N|} \ln \Xi^{\xi}_{\mathbb{G}_N}(q) - \frac{1}{2}(\alpha_1\Delta_1 + \ldots + \alpha_k\Delta_k)\ln(1-p) + \ln q$$

Thus in order to show that the pressure exists we need to prove that the limit

$$\Pi_{\mathbb{G}}(p,q) = \lim_{N\to\infty} \frac{1}{|V_N|} \ln \Xi^{0}_{\mathbb{G}_N}(q) \tag{4.33}$$

exists, is independent of V_N and has a finite radius of convergence.
By the previous analysis, when the condition (4.29) is satisfied, the logarithm of $\Xi^0_{\mathbb{G}_N}(p,q)$ converges absolutely, and we can use as an estimate of its radius of convergence ε^*_p instead of ε_p, since we are using for the computation of the pressure free boundary conditions. This ends the proof of theorem 4.2. □

5 The supercritical phase

5.1 More definitions about graphs and the main results in the supercritical regime

In order to study the supercritical phase we need to introduce the concept of cut-sets and minimal cut-sets of a graph. We will define a special class of minimal cut-sets in an infinite graph which may be regarded as the generalization of the concept of Peierls contours used in the Potts model defined in $\mathbb{Z}^d$. We recall that a *cut-set* of a graph $\mathbb{G} \in \mathcal{G}$ is a set $\gamma \subset \mathbb{E}$ such that the graph $(\mathbb{V}, \mathbb{E} - \gamma)$ is disconnected.

Definition 5.1 *A finite cut-set γ of an infinite connected graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{G}$ is called a fence if $(\mathbb{V}, \mathbb{E} - \gamma)$ has one and only one finite connected component and for all edges $e \in \gamma$ the graph $(\mathbb{V}, \mathbb{E} - (\gamma - e))$ has no finite connected component. If γ is a fence, we denote by $g_\gamma = (I_\gamma, E_\gamma)$ the unique finite connected component of $(\mathbb{V}, \mathbb{E} - \gamma)$. The set $I_\gamma \subset \mathbb{V}$ is called the vertex interior of the fence γ, and $O_\gamma = \mathbb{V} - I_\gamma$ is called the vertex exterior of the fence γ. Analogously the set $E_\gamma \subset \mathbb{E}$ is called the edge interior of the fence γ, and $\mathbb{E}_\gamma = \mathbb{E} - \{\gamma \cup E_\gamma\}$ is called the edge exterior of the fence γ.*

Note that for any fence γ of $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ it follows directly from the definition that $I_\gamma \cap O_\gamma = \emptyset$ and $I_\gamma \cup O_\gamma = \mathbb{V}$. Moreover $\gamma \cap E_\gamma = \gamma \cap \mathbb{E}_\gamma = E_\gamma \cap \mathbb{E}_\gamma = \emptyset$ and $E_\gamma \cup \gamma \cup \mathbb{E}_\gamma = \mathbb{E}$. From definition 5.1 it also follows that $\partial_e I_\gamma = \gamma$, $E_\gamma = \mathbb{E}|_{I_\gamma}$ and $\mathbb{E}_\gamma = \mathbb{E}|_{O_\gamma}$. Moreover, any edge $e \in \gamma$ is such that $e = \{x, y\}$ with $x \in I_\gamma$ and $y \in O_\gamma$. If $\gamma \subset \mathbb{E}$ is a fence, we put $\mathbb{G}_\gamma = (O_\gamma, \mathbb{E}_\gamma)$. Note that $\mathbb{G}_\gamma$ is an infinite graph but in general it is not connected. We finally denote by $\Gamma_{\mathbb{G}}$ the set of all fences in $\mathbb{G}$.
A slightly less immediate property of fences is given by the following proposition which shows that a fence γ is, $\forall v \in I_\gamma$, a (v, ∞)-minimal cut-set in the sense of [2].

Proposition 5.2 *Let γ be a fence in $\mathbb{G}$ and let $x \in I_\gamma$, then for any ray $\rho = (V_\rho, E_\rho)$ in $\mathbb{G}$ starting at x we have that $E_\rho \cap \gamma \neq \emptyset$.*

Proof. Suppose by contradiction that $E_\rho \cap \gamma = \emptyset$. Then $E_\rho \subset E^1_\rho \cup E^2_\rho$ with $E^1_\rho \subset E_\gamma$ and $E^2_\rho \subset \tilde{\mathbb{E}}_\gamma$ where $\tilde{\mathbb{G}}_\gamma = (\tilde{O}_\gamma, \tilde{\mathbb{E}}_\gamma)$ is some (infinite) connected component of $\mathbb{G}_\gamma$. The case $E^2_\rho = \emptyset$ would imply that $E_\rho \subset E_\gamma$ which is impossible since E_ρ is infinite and E_γ is finite. The case $E^1_\rho = \emptyset$ is impossible since no edge in $\mathbb{E}_\gamma$ has x as one of its end-points. Finally the last case $E^1_\rho \neq \emptyset$ and $E^2_\rho \neq \emptyset$ is impossible since otherwise $g_\gamma \cup \tilde{\mathbb{G}}_\gamma \subset (\mathbb{V}, \mathbb{E} - \gamma)$ would be connected and infinite which contradicts definition 5.1. $\square$

We will also use the following definitions:

Definition 5.3 *Given a fence $\gamma \subset \mathbb{E}$ and a vertex set $X \subset \mathbb{V}$, we say that γ surrounds X and we write $\gamma \bigodot X$ if $X \subset I_\gamma$. We say that γ separates X and we write $\gamma \bigotimes X$, if for any animal $a = (V_a, E_a)$ such that $X \subset V_a$, then $E_a \cap \gamma \neq \emptyset$.*

Definition 5.4 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{G}$, let $V \subset \mathbb{V}$ and let $R \geq 1$. We define the graph $\mathbb{G}|^R_V$ as the graph with vertex set V and edge set $E = \{\{x, y\} : x, y \in V$ and $d_{\mathbb{G}}(x, y) \leq R\}$. $V \subset \mathbb{V}$ is called R-connected if $\mathbb{G}|^R_V$ is connected. Analogously a set $S \subset \mathbb{E}$ is R-connected if its support V_S is R-connected.*

In other words a set $V \subset \mathbb{V}$ is R-connected in $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, if for any partition $\{A, B\}$ of V such that $A \cap B = \emptyset$ and $A \cup B = V$ we have that $d_{\mathbb{G}}(A, B) \leq R$.

Definition 5.5 *A graph $\mathbb{G} \in \mathcal{G}$ is called cut-set-bounded if there exists $R < +\infty$ such that every fence γ in $\mathbb{G}$ is R-connected. We denote by $\mathcal{P}$ the subclass of $\mathcal{G}$ of all cut-set-bounded graphs. Given a cut-set-bounded graph $\mathbb{G}$ we call the constant*

$$R_{\mathbb{G}} = \min\{R \in \mathbb{R} : \textit{ every cut-set is } R \textit{ connected}\} \tag{5.1}$$

the cut-set constant of $\mathbb{G}$.

Definition 5.6 . *Let $\mathbb{G}$ be locally finite graph, and let, for any $n \in \mathbb{N}$*

$$\mathcal{W}_n = \{W \subset \mathbb{V} : |W| < \infty\,,\ W \text{ connected}, \operatorname{diam}(W) = n\}$$

We define the function $f_{\mathbb{G}} : \mathbb{N} \to \mathbb{N}$ with

$$f_{\mathbb{G}}(n) = \min_{W \in \mathcal{W}_n} |\partial W| \tag{5.2}$$

so that

$$|\partial_e W| \geq f_{\mathbb{G}}(\operatorname{diam}(W)) \quad \text{for all } W \subset \mathbb{V} \text{ finite and connected} \qquad (5.3).$$

The function $f_{\mathbb{G}}$ is called the cut-set function of the graph.

Roughly speaking, this function measure how, in a graph $\mathbb{G}$, the boundary of connected sets of minimal boundary grows with the diameter of the set. Note that, by definition, $f_{\mathbb{G}}$ grows at most linearly with n in any bounded degree graph. Indeed, for most of the known examples (e.g. $\mathbb{Z}^d$ and regular trees) $f_{\mathbb{G}}$ is a linear function. To construct an example of $\mathbb{G}$ for which $f_{\mathbb{G}}$ grows slower than linearly, e.g. as $\ln n$, consider the infinite subset of $\mathbb{Z}^2$ below the curve $\ln x$ and above the x-axis. It is not difficult to see that such a graph has sets of diameter n that can be disconnected from the graph by deleting $\ln n$ edges.

Definition 5.7 *An infinite graph $\mathbb{G}$ is called a percolative graph if $\mathbb{G} \in \mathcal{P} \cap \mathcal{B}$ and its cut-set function $f_{\mathbb{G}}$ admits the lower bound*

$$f_{\mathbb{G}}(n) \geq C \ln n \tag{5.4}$$

for some constant C. We denote by $\mathcal{L}$ the set of percolative graphs.

We refer to graphs satisfying definition above as percolative because, as we will see below, the conditions in definition 5.7 are sufficient conditions for a graph to exhibit a non trivial percolation threshold. Heuristically, the requirement that the graph belongs to the class $\mathcal{P} \cap \mathcal{B}$ is a sufficient condition for the number of fences (i.e. the analogous of contours of the Ising model in $\mathbb{Z}^d$) of size n containing a fixed edge to grow at most as C^n, while the condition (5.4) is enough to guarantee that the number of possible positions of fences of size n surrounding a fixed vertex can be at most C^n (which occurs when $f_{\mathbb{G}} \sim \ln n$). We remark that our conditions are far from being necessary. For example, the class of graphs $\mathcal{P} \cap \mathcal{B}$ does not contain the trees (trees have fences which are not R-connected for any finite R) which do exhibit a non trivial percolation threshold.

To study the infinite volume limit of the connectivity functions in percolative graphs and in particular to ensure independence of this limit from boundary conditions $\xi = 0, 1$, we will need to slightly restrict the class of sequence $\{V_N\}$ along which this limit is taken. So we have to introduce one more definition.

Definition 5.8 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{P}$ with cut-set function $f_{\mathbb{G}}(n)$ and let $\{V_N\}$ a sequence of subsets of $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$; we say that V_N is a cut-set bounded sequence if for all N and for all fences γ such that $V_N \cap I_\gamma \neq \emptyset$, we have that the edge set $\gamma \cap \mathbb{E}_N$ is R-connected where R is the cut-set constant of $\mathbb{G}$.*

We were not able to find a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{P}$ which does not admit a cut-set-bounded sequence of sets V_N invading $\mathbb{V}$. Roughly speaking one should be able to produce example of graphs $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ in $\mathcal{P}$ with cut-set constant R such that, given any finite set $V \subset \mathbb{V}$ there are fences γ of $\mathbb{G}$ such that $\gamma \cap \mathbb{E}|_V$ is not R-connected. On the other hand, we were also not able to prove that if $\mathbb{G} \in \mathcal{P}$ then it always exists such a sequence.
We are now in the position to state our results concerning the supercritical regime of the Random Cluster model with free or wired boundary conditions and for p sufficiently close to 1. These results will be resumed by stating two theorems, the first concerning the finite connectivity functions and the second concerning the pressure. We remind that in the supercritical phase the interesting quantities are the *finite* connectivity functions (see comments after definition 3.5 and, for $q = 1$, see also [14]) defined in (3.6). That is why the theorem 5.9 below will be stated in term of these quantities.

Theorem 5.9 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{L}$ with cut-set constant R, let $\{V_N\}_{N\in\mathbb{N}}$ be any cut-set bounded sequence in $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$, let $q > 0$ be fixed, and let $(1-p)$ so small that $eA(1+\Delta^{R+1})\delta_p \leq 1$ where Δ is the maximum degree of $\mathbb{G}$ and*

$$A = \Big[\max\{2C, 1\}\Big] \times \Big[\Delta^{2R}\Big] \tag{5.5}$$

$$\delta_p = \max\left\{ \Big|\frac{1-p}{p}\Big| q, \Big|\frac{1-p}{p}\Big| \right\} \tag{5.6}$$

Then:
i) the infinite volume connectivity functions of the RCM on $\mathbb{G}$ with free and wired boundary conditions, defined in the limit (3.5), exist and are both equal to a function $\phi^{\rm f}_{p,q}(X)$ which can be written explicitly in term of an absolutely convergent series analytic as a function of p near 1, and does not depend on the sequence V_N.

ii) $|\phi^{\rm f}_{p,q}(X)|$ admits the bound

$$|\phi^{\rm f}_{p,q}(X)| \;\leq\; (1 + \Delta^{-R-1})(Ae\, \delta_p\,)^{f_{\mathbb{G}}(\mathrm{diam}\, X)}$$

where C is the constant appearing in (5.4) and $f_{\mathbb{G}}$ the monotonic function defined in (5.2) (definition 5.6).

Remark 5.10 *The theorem 5.9 implies that the percolation probability $\theta_{p,q}(x_0 \leftrightarrow \infty)$ is analytic in p and is of the order $1-(1-p)^{\Delta}$ uniformly in x_0, since $\theta_{p,q}(x_0 \leftrightarrow \infty) = 1 - \phi^{\rm f}_{p,q}(x_0)$. In other words, the random cluster model on percolative graphs has a percolation probability threshold p_c strictly less than 1. On the other hand theorem 4.1 immediately implies that that $p_c > 0$ in any bounded degree graph, and since any percolative graph is bounded degree, we have immediately the corollary below, which can be considered as a generalization, for values of $0 < q < 1$ and for percolative graphs, of theorem 4.2 in [1] stated for $\mathbb{G} = \mathbb{Z}^d$ and $q \geq 1$.*

Corollary 5.11 *Let $\mathbb{G}$ be an infinite graph and consider the random cluster model on $\mathbb{G}$ with free or wired boundary conditions. Then, if $\mathbb{G} \in \mathcal{L}$, for any $q > 0$, the critical percolation probability defined in (3.10) is such that $p^{\xi}_{c}(q) < 1$, with $\xi = 0, 1$.*

We remark that, due to the lack of validity of FKG inequalities, in the region $q < 1$ we cannot conclude that the percolation probability is monotonic increasing with p, so in principle in this region cannot be excluded the possibility of more than one critical point.

Remark 5.12 *The theorem 5.9 also suggests that the fall-off rate of the finite connectivity functions at large distances in general graphs in the highly supercritical phase may not necessarily be exponential, depending on the behavior of the function $f_{\mathbb{G}}$ defined in (5.2). In particular, for graphs such that $f_{\mathbb{G}}(n) \approx C \ln n$ it seems reasonable to conjecture that the finite connectivity functions decay polynomially. We plan to prove such claim (searching for a lower bound on the finite connectivities) in a future paper at least for $q = 1$ (i.e. Bernoulli percolation) where calculations are much simpler.*

We now state the second theorem concerning the pressure.

Theorem 5.13 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E}) \in \mathcal{L} \cap \mathcal{A} \cap \mathcal{Q}^v \cap \mathcal{Q}^e$, let $\{V_N\}_{N\in\mathbb{N}}$ be any Følner sequence in $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$, and let $(1-p)$ so small that $eA(1 + \Delta^{R+1})\delta_p \leq 1$ where δ_p is defined in (5.6). Then the pressure of Random Cluster Model on $\mathbb{G}$, defined in (3.4) exists and can be written explicitly in term of an absolutely convergent series which is analytic as a function of p, and does not depend on V_N and on ξ.*

5.2 Proof of theorem 5.9. Polymer expansion for the finite connectivity functions

In this section we will assume that $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is percolative with maximum degree Δ , with cut-set constant R and with cut-set function $f_{\mathbb{G}}$. We will also assume that $\{V_N\}$ is a cut-set bounded sequence in $\mathbb{G}$ such that $V_N \nearrow \mathbb{V}$.
The finite volume free and wired finite connectivity functions for any $X \subset V_N - \partial_v^{\rm int} V_N$ can be written as

$$\phi^{{\rm f},N}_{p,q,\xi}(X) = \frac{1}{\bar{Z}^{\xi}_N(p,q)} \sum_{\substack{\omega\in\Omega^{\xi}_{\mathbb{G}_N}:\ \exists g\in A_{\mathbb{G}}:\ E_g\subset O(\omega)\\ X\subset V_g,\ V_g\cap\ \partial_v^{\rm int} V_N=\emptyset}} \lambda^{|C(\omega_N)|} q^{k^{\xi}_N(\omega)} \tag{5.7}$$

where in this section

$$\lambda = \frac{1-p}{p}$$

and

$$\bar{Z}^{\xi}_N(p,q) = \sum_{\omega\in\Omega^{\xi}_{\mathbb{G}_N}} \lambda^{|C(\omega_N)|} q^{k^{\xi}_N(\omega)} = p^{|\mathbb{E}_N|} Z^{\xi}_{\mathbb{G}_N}(p,q) \tag{5.8}$$

We recall that the symbol $C(\omega_N)$ denotes the set of closed edges in $\mathbb{E}_N$ once the configuration $\omega \in \Omega^{\xi}_{\mathbb{G}_N}$ is given.

Definition 5.14 *A subset $S \subset \mathbb{E}$ is called a dual animal if it is finite and it is R-connected. We say that two dual animals S and S' are compatible and we write $S \sim S'$ if $S \cup S'$ is not a dual animal (i.e. $d_{\mathbb{G}}(S, S') > R$). We will denote by $\mathcal{E}_{\mathbb{G}}$ the set of all dual animals in $\mathbb{E}$. We will also denote by $\mathcal{E}_N$ the set of dual animals in $\mathbb{E}_N$.*

Observe that, since $\mathbb{G}$ is assumed to be cut-set bounded, every fence in $\mathbb{G}$ is a dual animal.

Definition 5.15 *Let $S \subset \mathbb{E}$ and let $\gamma \subset S$ be a fence with vertex interior V_γ and edge interior E_γ. We say that γ is minimal with respect to S if there is no other fences $\gamma' \subset S$ such that $\gamma' \cap \gamma \neq \emptyset$ and $\gamma' \subset \gamma \cup E_\gamma$ (recall: E_γ is the edge interior of γ). Note that a minimal fence γ can contain in its interior a fence γ' such that $\gamma \cap \gamma' = \emptyset$. Given $S \subset \mathbb{E}$ we denote by n_S the number of fences which are minimal with respect to S.*

Remark 5.16 *By the definition above and by definition 5.1, if $S \subset \mathbb{E}$ is finite, then the number of finite connected component of $(\mathbb{V}, \mathbb{E} - S)$ is exactly n_S.*

We will now give convenient expressions for $k^0_N(\omega)$ and $k^1_N(\omega)$. Let us consider first the case $k^1_N(\omega)$ which is the easier one. If we are using wired boundary conditions, then $k^1_N(\omega)$ is the number of connected components of $O(\omega_N)$ plus the isolated vertices whose support is contained in $V_N^{\rm int}$. The fences associated with any of such components is then totally contained in $\mathbb{E}_N$. This means that

$$k^1_N(\omega) = n_{C(\omega_N)} \tag{5.9}$$

Using now (5.9) the partition function $\bar{Z}^1_{\mathbb{G}_N}(p,q)$ defined in (5.8) can be rewritten as

$$\bar{Z}^1_N(p,q) = \sum_{\omega \in \Omega^1_{\mathbb{G}_N}} \lambda^{|C(\omega_N)|} q^{k^1_N(\omega)} = \sum_{\omega \in \Omega^1_{\mathbb{G}_N}} \lambda^{|C(\omega_N)|} q^{n_{C(\omega_N)}} \tag{5.10}$$

and

$$\phi^{{\rm f},N}_{p,q,1}(X) = \frac{1}{\bar{Z}^1_{\mathbb{G}|_V}(p,q)} \sum_{\substack{\omega \in \Omega^1_{\mathbb{G}_N}:\ \exists g \in A_{\mathbb{G}}:\ E_g \in O(\omega) \\ X \subset V_g,\ V_g \cap\ \partial_v^{\rm int} V_N = \emptyset}} \lambda^{|C(\omega_N)|} q^{n_{C(\omega_N)}}$$

The case $k^0_N(\omega)$ is more involved. Observe first that the term in the partition function

$$\bar{Z}^0_N(p,q) = \sum_{\omega \in \Omega^0_{\mathbb{G}_N}} \lambda^{|C(\omega_N)|} q^{k^0_N(\omega)}$$

corresponding to the configuration in which all bonds are open is q (since $k^0_N(\omega) = 1$ in this case). For technical reasons is convenient that this term is 1 (as it is in $\bar{Z}^0_N(p,q)$). So we define

$$\hat{Z}^0_N(p,q) = \sum_{\omega \in \Omega^0_{\mathbb{G}_N}} \lambda^{|C(\omega_N)|} q^{k^0_N(\omega)-1} \tag{5.11}$$

whence

$$q\hat{Z}^0_N(p,q) = \bar{Z}^0_N(p,q) \tag{5.12}$$

in such a way that $\hat{Z}^0_N(p,q)$ can be interpreted as a partition function with term equal to 1 corresponding to the configuration in which all edges are open.
Now, by definition we can write

$$\phi^{{\rm f},N}_{p,q,0}(X) = \frac{1}{\hat{Z}^0_N(p,q)} \sum_{\substack{\omega \in \Omega^{\xi}_{\mathbb{G}_N}:\ \exists g \in A_{\mathbb{G}}:\ E_g \in O(\omega) \\ X \subset V_g,\ V_g \cap\ \partial_v^{\rm int} V_N = \emptyset}} \lambda^{|C(\omega_N)|} q^{k^0_N(\omega)-1}$$

We have now to write the explicit expression of $k_N^0(\omega)$. In this case we have to count the fences in the set $C(\omega_N)\cup\partial_e V_N \equiv \bar{C}(\omega_N)$, and therefore we allow fences $\bar\gamma$ such that $\bar\gamma\cap\partial_e V_N \neq \emptyset$; in the latter case the set $g \equiv \bar\gamma - \partial_e V_N$ will be called from now on *wall*. Observe that since V_N is a cut-set bounded sequence (see definition 5.8), then a wall in $\mathbb{E}_N$ is R-connected, i.e. is a dual animal.
The number $k_N^0(\omega)$ is then simply

$$k_N^0(\omega) = n_{\bar{C}(\omega_N)}$$

Let us define for a given $S \in \mathcal{E}_N$

$$\tilde n_S = \begin{cases} n_S & \text{if } S\cup\partial_e V_N \notin \mathcal{E} \\ \\ n_{S\cup\partial_e V_N} - 1 & \text{if } S\cup\partial_e V_N \in \mathcal{E} \end{cases} \tag{5.13}$$

and its activity $\rho^\xi(S)$ as follows

$$\rho^\xi(S) = \begin{cases} \lambda^{|S|} q^{n_S} & \text{if } \xi = 1 \\ \\ \lambda^{|S|} q^{\tilde n_S} & \text{if } \xi = 0 \end{cases} \tag{5.14}$$

Defining

$$\delta_p = \max\{(|\lambda| q), |\lambda|\} \tag{5.15}$$

We have

$$|\rho^\xi(S)| \le \delta_p^{|S|}, \tag{5.16}$$

The reason why we need to define for free boundary conditions the quantity $\tilde n_S$ is the following: for a fixed dual animal containing a wall, we can obtain a fence from the union of the wall and the (closed) boundary in two different ways, while we want to count the unit increasing of the number of connected components of the configuration. This is the reason of the -1 in the definition of $\tilde n_S$.
Furthermore, define the hard core pair potential between two dual animals S_i, S_j as

$$U(S_i,S_j) = \begin{cases} +\infty & \text{if } S_i \not\sim S_j \\ \\ 0 & \text{otherwise,} \end{cases} \tag{5.17}$$

Use the shorthand notations

$$\mathbf{S}_n = (S_1,\dots,S_n)\ ; \quad \rho^\xi(\mathbf{S}_n) \equiv \rho^\xi(S_1)\cdots\rho^\xi(S_n); \quad U(\mathbf{S}_n) = \sum_{1\le i<j\le n} U(S_i,S_j)$$

Then define the ξ dependent (for $\xi = 0,1$) polymer gas partition function as

$$\Psi_N^\xi(p,q) = 1 + \sum_{n\ge 1}\frac{1}{n!}\sum_{\mathbf{S}_n\in(\mathcal{E}_N)^n} \rho^\xi(\mathbf{S}_n) e^{-U(\mathbf{S}_n)} \tag{5.18}$$

where $(\mathcal{E}_N)^n$ is the n-times cartesian product of $\mathcal{E}_N$. Note that, by construction

$$\Psi_N^1(p,q) = \bar Z_N^1(p,q), \qquad \Psi_N^0(p,q) = \hat Z_N^0(p,q) \tag{5.19}$$

and also

$$\phi^{\mathrm{f}}_{p,q,\xi}(X) = \frac{1}{\Psi^{\xi}_{N}(p,q)} \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{S}_n \in (\mathcal{E}_N)^n \\ \mathbf{S}_n \odot X}} \rho^{\xi}(\mathbf{S}_n) e^{-U(\mathbf{S}_n)} \tag{5.20}$$

where condition $\mathbf{S}_n \odot X$ on the sum above means that there must exist a fence $\gamma \subset \cup_{i=1}^{n} S_i$ such that $\gamma \odot X$ and the set $\bar{E}_\gamma \cap [\cup_{i=1}^{n} S_i]$ does not contains fences γ' such that $\gamma' \otimes X$ (here $\bar{E}_\gamma = \gamma \cup E_\gamma$).
We now rewrite the ratio (5.20) (between two finite sums) as a series. We follow and generalize the ideas developed in [7] and [8] for $\mathbb{Z}^d$. So we will define objects more general than dual animals which will be called polymers.

Definition 5.17 *Let $X \subset \mathbb{V}$ finite, a set $P \subset \mathbb{E}$ is called X-R-connected if $P = \cup_{i=1}^{k} S_i$ with $k \geq 1$ and the following holds: for all $i = 1, 2, \ldots, k$ $S_i \in \mathcal{E}_{\mathbb{G}}$; for all $i, j = 1, 2, \ldots, k$, $S_i \sim S_j$ and each S_i contains a fence γ_i such that $\gamma_i \odot Y$ for some non empty $Y \subset X$.*

We will denote by Π^X the set of all X-R-connected sets in $\mathbb{E}$ and by Π^X_N the set of all X-R-connected sets in $\mathbb{E}_N$. We will also put $\mathcal{E}^X_{\mathbb{G}} = \mathcal{E}_{\mathbb{G}} \cup \Pi^X$ and $\mathcal{E}^X_N = \mathcal{E}_N \cup \Pi^X_N$.

Definition 5.18 *A set $P \in \mathcal{E}^X_{\mathbb{G}}$ will be called a X-polymer (or simply polymer when it is clear from the contest). We will say that two polymers $P_i \in \mathcal{E}^X_{\mathbb{G}}$ and $P_j \in \mathcal{E}^X_{\mathbb{G}}$ are compatible, and we write $P_i \approx P_j$, if $P_i \cup P_j \notin \mathcal{E}^X_{\mathbb{G}}$; viceversa, $P_i \in \mathcal{E}^X_{\mathbb{G}}$ and $P_j \in \mathcal{E}^X_{\mathbb{G}}$ are incompatible, and we write $P_i \not\approx P_j$, if $P_i \cup P_j \in \mathcal{E}^X_{\mathbb{G}}$.*

Note that if $P \in \Pi^X$ and $P' \in \Pi^X$ then necessarily $P \not\approx P'$.
If $P \in \Pi^X$ and $P = \cup_{i=1}^{k} S_i$ with $k \geq 2$ we define the activity of the polymer P as $\rho^{\xi}(P) = \prod_{i=1}^{k} \rho^{\xi}(S_i)$. Define further the hard core pair potential between two polymers P_i, P_j as

$$\tilde{U}(P_i, P_j) = \begin{cases} +\infty & \text{if } P_i \not\approx P_j \\ \\ 0 & \text{otherwise,} \end{cases} \tag{5.21}$$

Again, we use the shorthand notations

$$\mathbf{P}_n = (P_1, \ldots, P_n)\ ; \quad \rho^{\xi}(\mathbf{P}_n) \equiv \rho^{\xi}(P_1) \cdots \rho^{\xi}(P_n); \quad \tilde{U}(\mathbf{P}_n) = \sum_{1 \leq i < j \leq n} \tilde{U}(P_i, P_j)$$

Then, the r.h.s. of (5.20) can be rewritten as

$$\phi^{\mathrm{f},N}_{p,q,\xi}(X) = \frac{1}{\Psi^{\xi}_{N}(p,q)} \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_N)^n \\ \exists ! i \in \mathrm{I}_n :\ P_i \odot X}} \rho^{\xi}(\mathbf{P}_n) e^{-\tilde{U}(\mathbf{P}_n)} \tag{5.22}$$

and the partition function can be rewritten as

$$\Psi^{\xi}_{N}(p,q) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\mathbf{P}_n \in (\mathcal{E}^X_N)^n} \rho^{\xi}(\mathbf{P}_n) e^{-\tilde{U}(\mathbf{P}_n)}$$

Analogously as we did in section 4, we define, for $P \in \mathcal{E}^X$

$$\Pi^{\mathrm{f},N}_{p,q,\xi}(P) = \frac{\partial}{\partial\rho^{\xi}(P)} \ln\left[\Psi^{\xi}_{N}(p,q)\right]$$

Then, by construction

$$\phi^{\mathrm{f},N}_{p,q,\xi}(X) = \sum_{\substack{P\in\mathcal{E}^X_N\\ P\odot X}} \rho^{\xi}(P)\Pi^{\mathrm{f},N}_{p,q,\xi}(P) \tag{5.23}$$

Now, by standard cluster expansion it is well known that

$$\ln\Psi^{\xi}_{N}(p,q) = \sum_{n\geq 1}\frac{1}{n!}\sum_{\mathbf{P}_n\in(\mathcal{E}^X_N)^n} \Phi^T(\mathbf{P}_n)\rho^{\xi}(\mathbf{P}_n) \tag{5.24}$$

where the Ursell coefficients $\Phi^T(\mathbf{P}_n)$ are given by

$$\Phi^T(\mathbf{P}_n) = \begin{cases} \sum\limits_{\substack{E\subset E(\mathbf{P}_n)\\ (\mathrm{I}_n,E)\in\mathcal{G}_n}} (-1)^{|E|} & \text{if } n\geq 2\\ \\ 1 & \text{if } n=1. \end{cases} \tag{5.25}$$

where $E(\mathbf{P}_n) = \{\{i,j\}\subset \mathrm{I}_n : P_i \not\approx P_j\}$ and $\mathcal{G}_n$ denotes the set of all connected graphs with vertex set I_n. So

$$\Pi^{\mathrm{f},N}_{p,q,\xi}(P) = \sum_{n\geq 0}\frac{1}{n!}\sum_{\mathbf{P}_n\in(\mathcal{E}^X_N)^n} \Phi^T(P,\mathbf{P}_n)\rho^{\xi}(\mathbf{P}_n) \tag{5.26}$$

We also define functions on the whole $\mathbb{G}$ (hence not depending on boundary conditions) as follows

$$\Pi^{\mathrm{f}}_{p,q,}(P) = \sum_{n\geq 0}\frac{1}{n!}\sum_{\mathbf{P}_n\in(\mathcal{E}^X_{\mathbb{G}})^n} \Phi^T(P,\mathbf{P}_n)\rho^{\xi}(\mathbf{P}_n) \tag{5.27}$$

and

$$\phi^{\mathrm{f}}_{p,q}(X) = \sum_{\substack{P\in\mathcal{E}^X_{\mathbb{G}}\\ P\odot X}} \rho(P)\Pi^{\mathrm{f}}_{p,q,}(P) \tag{5.28}$$

which, as we will see, represents an absolutely convergent expansion for p near 1 for the infinite volume finite connectivity function.

5.3 Proof of theorem 5.9. Convergence of the finite connectivity functions

As we did in section 4, we first prove an exponential bound on the activity $\rho(R)$.

Lemma 5.19 *Let $\mathbb{G}$ be a cut-set bounded and bounded degree graph. Then for any $n\geq 1$*

$$\sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}\\ e\in S,\ |S|=n}} 1 + \sum_{\substack{P\in\Pi^X\\ |P|=n}} 1 \quad\leq\quad A^n \tag{5.29}$$

where

$$A = \left[\max\{2C,1\}\right]\times\left[\Delta^{2R}\right] \tag{5.30}$$

with C being the constant appearing in (5.4)

Proof. We start bounding the first term in r.h.s. of (5.29) i.e. the number of dual animals of fixed cardinality containing a fixed edge. We recall that a dual animal is just a R-connected set of $\mathbb{E}$. Thus recalling definition 5.5 we have

$$\sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}\\ e\in S,\ |S|=n}} 1 \le \sup_{e\in\mathbb{E}} \sum_{\substack{S\subset\mathbb{E}:\ S \text{ connected}\\ e\in S,\ |S|=Rn}} 1 \le \Delta^{2Rn} \tag{5.31}$$

Concerning the second term in l.h.s. of (5.29) this sum is done only over Polymers P of the form $P=\cup_{i=1}^m S_i$ with $m\ge 1$ such that, for all $i=1,2,\dots,m$: $S_i\in\mathcal{E}_{\mathbb{G}}$; for all $i,j=1,2,\dots,m$, $S_i\sim S_j$; and each S_i contains a fence γ_i such that $\gamma_i \odot Y$ for some $Y\subset X$. Hence

$$\sum_{P\in\Pi^X:|P|=n} 1 \le \sum_{m=1}^{n} \sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} \left[\sup_{x\in\mathbb{V}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}:\ |S|=k_i\\ \exists\gamma\subset S:\,\gamma\odot x}} 1 \right]$$

Now, to bound the factor

$$\sup_{x\in\mathbb{V}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}:\ |S|=k_i\\ \exists\gamma\subset S:\,\gamma\odot x}} 1$$

we proceed as follows. Since $\mathbb{G}$ is connected and locally finite, for any $x\in\mathbb{V}$ there exists a geodesic ray $\rho=(V_\rho,E_\rho)$ starting at x. Then, since S must contain a fence γ such that $\gamma\odot x$, we have, by proposition 2.2, that $E_\rho\cap\gamma\neq\emptyset$. Let $e_x(\gamma)$ be the first edge (in the natural order of the ray) in E_ρ which belongs to γ and define

$$r_{k_i}(x)=\{e\in E_\rho : \exists\gamma\in\Gamma_{\mathbb{G}} \text{ such that } |\gamma|=k_i \text{ and } e=e_x(\gamma)\} \tag{5.32}$$

Hence

$$\sup_{x\in\mathbb{V}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}:\ |S|=k_i\\ \exists\gamma\subset S:\,\gamma\odot x}} 1 = \sup_{x\in\mathbb{V}} \sum_{e\in r_{k_i}(x)} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}\ |S|=k_i\\ \exists\gamma\subset S:\,\gamma\odot x\\ e_x(\gamma)=e}} 1 \le \sup_{x\in\mathbb{V}} |r_{k_i}(x)| \sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}\\ e\in S,\ |S|=k_i}} 1$$

Now we observe that the interior I_γ of γ is a finite and connected subset of $\mathbb{V}$ and recalling the definition of the diameter (2.4) we have clearly that

$$\sup_{x\in\mathbb{V}} |r_n(x)| \le \sup_{x\in\mathbb{V}} \sup_{\substack{\gamma\in\Gamma_{\mathbb{G}}:\,\gamma\odot x\\ |\gamma|=n}} \mathrm{diam} I_\gamma$$

But, by (5.3) and (5.4), we have immediately that $\mathrm{diam} I_\gamma \le C^n$ so we get that $\sup_{x\in\mathbb{V}}|r_n(x)|\le C^n$. Hence, recalling (5.31)

$$\sup_{x\in\mathbb{V}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}:\ |S|=k_i\\ \exists\gamma\subset S:\,\gamma\odot x}} 1 \le \left[C\Delta^{2R}\right]^{k_i}$$

so the second term

$$\sum_{\substack{P\in\Pi^X\\ |P|=n}} 1 \le \sum_{m=1}^{n} \sum_{k_1+\ldots+k_m=n} \prod_{i=1}^{m} \left[C\Delta^{2R}\right]^{k_i} = \left[C\Delta^{2R}\right]^n \sum_{m=1}^{n} \sum_{k_1+\ldots+k_m=n} 1 \le \left[2C\Delta^{2R}\right]^n$$

□

We now prove the following lemma

Lemma 5.20 *For any $q > 0$ the function $\phi^{\mathrm{f}}_{p,q}(X)$ defined by (5.23) is analytic as a function of p whenever $eA(1 + \Delta^{R+1})\varepsilon_p \leq 1$ where ε_p is defined in (5.15). Moreover $\phi^{\mathrm{f}}_{p,q}(X)$ satisfies the following bounds.*

$$|\phi^{\mathrm{f}}_{p,q}(X)| \ \leq \ (1 + \Delta^{-R-1})(Ae\delta_p)^{f_{\mathbb{G}}(\operatorname{diam} X)}$$

where A is the constant defined in (5.30) and $f_{\mathbb{G}}$ is the monotonic function defined in definition 5.6. Moreover, if $\{V_N\}_{N\in\mathbb{N}}$ is any cut-set bounded sequence of subsets of $\mathbb{V}$, then, for all $N \in \mathbb{N}$ the function $\phi^{\mathrm{f},N}_{p,q,\xi}(X)$ defined by (5.7) is analytic as a function of p whenever $eA(1+\Delta^{R+1})\varepsilon_p \leq 1$.

Proof. We use here the Kotecky-Preiss condition [29], which in this case can be checked easily. We stress that our bounds are not optimal. So, the Kotecky-Preiss condition for the polymer gas with set of polmymers $P \in \mathcal{E}^X_{\mathbb{G}}$ and with activity $\rho^\xi(P)$ states that series (5.24), (5.26), (5.27) converge if it is possible to find $a > 0$ such that for all polymers $P' \in \mathcal{E}^X_{\mathbb{G}}$

$$\sum_{\substack{P\in\mathcal{E}^X_{\mathbb{G}} \\ P\not\approx P'}} |\rho^\xi(P)|e^{a|P|} \leq a|P'| \tag{5.33}$$

Recalling the estimate (5.16), one can easily check that (5.33) becomes

$$\sum_{n=1}^{\infty}(\delta_p e^a)^n \sum_{\substack{P\in\mathcal{E}^X_{\mathbb{G}}\ |P|=n \\ P\not\approx P'}} 1 \ \leq \ a|P'| \tag{5.34}$$

Now we have that

$$\sum_{\substack{P\in\mathcal{E}^X_{\mathbb{G}}:\, P\not\approx P' \\ |P|=n}} 1 \leq \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}:\, |S|=n \\ d_{\mathbb{G}}(S,P')\leq R}} 1 + \sum_{P\in\Pi^X:\, |P|=n} 1 \tag{5.35}$$

Now, let us define the edge set $B_R(P') = \{e \in \mathbb{E} : d_{\mathbb{G}}(e, P') \leq R\}$, then

$$\sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}:\, |S|=n \\ d_{\mathbb{G}}(S,P')\leq R}} 1 \leq |B_R(P')| \sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}} \\ e\in S,\ |S|=n}} 1$$

We bound $|B_R(P')|$. Let $B^v_R(P') = \{v \in \mathbb{V} : d_{\mathbb{G}}(v, P') \leq R\}$, then, since $\mathbb{G}$ has maximum degree Δ and since each edge in $\mathbb{E}$ is incident to two vertices in $\mathbb{V}$ we have surely that

$$|B_R(P')| \leq \frac{\Delta}{2}B^v_R(P') \leq \frac{\Delta}{2}\sum_{e\in P'} B^v_R(e) \leq \frac{\Delta}{2}|P'|\Delta^R \leq \Delta^{R+1}|P'|$$

Whence the first term in r.h.s. of (5.35) is bounded by

$$\sum_{\substack{S\in\mathcal{E}_{\mathbb{G}}:\, |S|=n \\ d_{\mathbb{G}}(S,P')\leq R}} 1 \leq \Delta^{R+1}|P'| \ \sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}} \\ e\in S,\ |S|=n}} 1$$

Hence, by lemma 5.19, we have that

$$\sum_{\substack{P\in\mathcal{E}^X_{\mathbb{G}}:\, P\not\approx P' \\ |P|=n}} 1 \leq \Delta^{R+1}|P'|\left[\sup_{e\in\mathbb{E}} \sum_{\substack{S\in\mathcal{E}_{\mathbb{G}} \\ e\in\gamma,\ |S|=n}} 1 + \sum_{P\in\Pi^X:\, |P|=n} 1\right] \leq \Delta^{R+1}|P'|A^n \tag{5.36}$$

Hence (5.34) becomes

$$\sum_{n=1}^{\infty}(\delta_p e^a)^n A^n \;\le\; \frac{a}{\Delta^{R+1}} \tag{5.37}$$

choosing, $a = 1$ we get that the series (5.27) is absolutely converegent whenever

$$\delta_p \le \frac{1}{eA(1+\Delta^{R+1})}$$

and it is bounded by

$$|\Pi^{\mathrm{f}}_{p,q,}(P)| \le e^{|P|}$$

Whence, recalling (5.28)

$$|\phi^{\mathrm{f}}_{p,q}(X)| \le \sum_{\substack{P\in\mathcal{E}^X_{\mathbb{G}}\\ P\odot X}} (e\delta_p)^{|P|}$$

Now let us find a lower bound for the number $\min_{P\bigodot X}|P|$.
Let U_X be a subset of $\mathbb{V}$ definite as follows. U_X is connected, $X \subset U_X$ and $|\partial_e U_X|$ is minimum, i.e if U is another connected subset of $\mathbb{V}$ such that $U \supset X$ then $|\partial_e U| \ge |\partial_e U_X|$. Now since $P \bigodot X$ then by construction that $|P| \ge |\partial_e U_X|$ since by definition P contains a fence with vertex interior containing X. Now, since $\partial_e U_X$ is a fence, then it is R-connected. This means that

$$|P| \ge |\partial_e U_X| \ge C\, f_{\mathbb{G}}(\operatorname{diam} U_X) \ge C\, f_{\mathbb{G}}(\operatorname{diam} X)$$

So, using also (5.29)

$$|\phi^{\mathrm{f}}_{p,q}(X)| \le \sum_{n\ge\frac{1}{R}\operatorname{diam} X} (e\delta_p)^n \sum_{\substack{P\in\mathcal{E}^X_{\mathbb{G}}:|P|=n\\ P\odot X}} 1 \le \sum_{n\ge\frac{1}{R}\operatorname{diam} X} (Ae\delta_p)^n \le (1+\Delta^{-R-1})(Ae\delta_p)^{f_{\mathbb{G}}(\operatorname{diam} X)}$$

The proof of the second part of the lemma, i.e. the analitycity of $\phi^{\mathrm{f},N}_{p,q,\xi}(X)$ can be done in a similar way by observing that $\phi^{\mathrm{f},N}_{p,q,\xi}(X)$ admits the polymer representation (5.22) analogous to (5.28) and $|\rho^{\xi}(P)| \le \delta_p$. $\square$

Now we prove the following lemma which concludes the proof of theorem 5.9

Lemma 5.21 *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a percolative graph and let $\{V_N\}$ be any cut-set bounded sequence in $\mathbb{V}$ such that $V_N \nearrow \mathbb{V}$. Then for any fixed $q > 0$ and p such that $eA(1+\Delta^{R+1})\delta_p \le 1$, and $\xi = 0, 1$*

$$\lim_{N\to\infty} \phi^{\mathrm{f},N}_{p,q,\xi}(X) = \phi^{\mathrm{f}}_{p,q}(X)$$

where $\phi^{\mathrm{f}}_{p,q}(X)$ is the function defined in (5.28).

Proof. We will consider only the case $\xi = 0$, which is the less trivial one.

$$|\phi^{\mathrm{f}}_{p,q}(X) - \phi^{\mathrm{f},N}_{p,q,\xi=0}(X)| \le |\sum_{\substack{P\in\mathcal{E}^X_{\mathbb{G}}\\ P\odot X}} \rho(P)\Pi^{\mathrm{f}}_{p,q,}(P) - \sum_{\substack{P\in\mathcal{E}^X_N\\ P\bigodot X}} \rho^{\xi}(P)\Pi^{\mathrm{f},N}_{p,q,\xi}(P)| \le$$

$$= \left| \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_{\mathbb{G}})^n \\ P_1 \odot X}} \Phi^T(\mathbf{P}_n)\rho(\mathbf{P}_n) \; - \; \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_N)^n \\ P_1 \odot X}} \Phi^T(\mathbf{P}_n)\rho^0(\mathbf{P}_n) \right| \leq$$

$$\leq \left| \sum_{n\geq 1} \frac{1}{(n-1)!} \left\{ \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_{\mathbb{G}})^n :\, P_1 \odot X \\ \exists j \in \mathrm{I}_n :\; P_j \not\subset \mathbb{E}_N}} \Phi^T(\mathbf{P}_n)\rho(\mathbf{P}_n) \; + \; \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_N)^n :\; P_1 \odot X \\ \exists j \in \mathrm{I}_n :\; P_j \text{ contains a wall}}} \Phi^T(\mathbf{P}_n) \Big[\rho(\mathbf{P}_n) - \rho^0(\mathbf{P}_n) \Big] \right\} \right| \leq$$

Using that $|\rho(\mathbf{P}_n) - \rho^0(\mathbf{P}_n)| \leq 2\delta_p^{\sum_{i=1}^n |P_i|}$, due to the bound (5.16), we get

$$\begin{aligned} |\phi^{\mathrm{f}}_{p,q}(X) - \phi^{\mathrm{f},N}_{p,q,\xi=0}(X)| \quad &\leq \quad \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_{\mathbb{G}})^n :\, P_1 \odot X \\ \exists j \in \mathrm{I}_n :\; P_j \not\subset \mathbb{E}_N}} \delta_p^{\sum_{i=1}^n |P_i|} |\Phi^T(\mathbf{P}_n)| \; + \\ &+ \quad 2 \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_N)^n :\; P_1 \odot X \\ \exists j \in \mathrm{I}_n :\; P_j \text{ contains a wall}}} \delta_p^{\sum_{i=1}^n |P_i|} |\Phi^T(\mathbf{P}_n)| \end{aligned} \tag{5.38}$$

Now, by lemma 5.20, we already know that for $eA(1+\Delta^{R+1})\delta_p \leq 1$ the two series in the left hand side of inequality (5.38) are analytic in δ_p. Consider the first term of the r.h.s. of (5.38). Let us split this term in two series as follows

$$\sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_{\mathbb{G}})^n :\, P_1 \odot X \\ \exists j \in \mathrm{I}_n :\; P_j \not\subset \mathbb{E}_N}} \delta_p^{\sum_{i=1}^n |P_i|} |\Phi^T(\mathbf{P}_n)| = A_1 + A_2$$

with

$$A_1 = \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_{\mathbb{G}})^n :\, P_1 \odot X \\ \exists j \in \mathrm{I}_n :\, ,\; P_j \not\subset \mathbb{E}_N ,\; P_i \neq P_j}} \delta_p^{\sum_{i=1}^n |P_i|} |\Phi^T(\mathbf{P}_n)|$$

$$A_2 = \sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_{\mathbb{G}})^n :\, P_1 \odot X \\ P_1 \not\subset \mathbb{E}_N}} \delta_p^{\sum_{i=1}^n |P_i|} |\Phi^T(\mathbf{P}_n)|$$

Analyticity of A_1 as a function of δ_p implies immediately that there exists a constant C_1 such that $\mathrm{C}_1\delta_p < 1$ and

$$A_1 \leq (\mathrm{C}_1\delta_p)^{n_0}$$

where the lowest order n_0 is

$$n_0 = \min_{\substack{\mathbf{P}_n \in (\mathcal{E}^X_{\mathbb{G}})^n \\ G(\mathbf{P}_n) \in \mathcal{G}_n ,\; P_1 \odot X \\ \exists j \in \mathrm{I}_n :\; P_j \not\subset \mathbb{E}_N ,\; P_i \neq P_j}} \{ \sum_{i=1}^n |P_i| \}$$

Here above the condition $G(\mathbf{P}_n) \in \mathcal{G}_n$ is due the presence the factor $\Phi^T(\mathbf{P}_n)$. It is easy to see that n_0 is at least

$$n_0 \geq \min_{\substack{\gamma\in\Gamma_{\mathbb{G}},\ \gamma \odot X,\\ S\in\mathcal{E}_{\mathbb{G}},\ S\not\subset\mathbb{E}_N\\ d_{\mathbb{G}}(\gamma,S)\leq R}} \{|\gamma| + |S|\}$$

Now, by (5.3) and (5.4), we have that $|\gamma| \geq \ln[\mathrm{diam}(I_\gamma)]$. So

$$n_0 \geq \ln\left[\min_{\substack{\gamma\in\Gamma_{\mathbb{G}},\ \gamma \odot X,\\ S\in\mathcal{E}_{\mathbb{G}},\ S\not\subset\mathbb{E}_N\\ d_{\mathbb{G}}(\gamma,S)\leq R}} \{\mathrm{diam}(I_\gamma) + |S|\}\right] \geq \ln\left[\min_{x\in X} \frac{1}{R} d_{\mathbb{G}}(x, \partial V_N)\right]$$

Thus, by lemma 4.7, the r.h.s. of inequality above is a divergent quantity when $N \to \infty$. So we have shown that $A_1 \to 0$ as $N \to \infty$. Concerning A_2 we have similarly

$$A_2 \leq \mathrm{Const}' \delta_p^{n_0'}$$

where now

$$n_0' = \min_{\substack{\mathbf{P}_n\in(\mathcal{E}^X_{\mathbb{G}})^n\\ P_1 \odot X,\, P_1\not\subset\mathbb{E}_N}} \{\sum_{i=1}^n |P_i|\} \ \geq \ \min_{\substack{P\odot X\\ P\not\subset\mathbb{E}_N}} \{|P|\}$$

this can be easily bounded from below as

$$n_0' \geq \min_{\substack{S\in\mathbb{E}_{\mathbb{G}}:\ S\odot X\\ S\not\subset\mathbb{E}_N}} \{|\gamma|\}$$

Similarly to the previous case, we have that the r.h.s. of the inequality above diverges when $N \to \infty$. $\square$

5.4 Proof of theorem 5.13.

In this section, accordingly to the hypothesis of theorem 5.13, we will assume that $\mathbb{G}$ is amenable and quasi-transitive and that the sequence $\{V_N\}_{N\in\mathbb{N}}$ is Følner.
By remark 3.2, if the pressure exists, it is independent of boundary conditions so we consider here the case $\xi = 1$ (wired boundary conditions) which is easier for p near 1.
Recalling (5.8), (5.10), (5.18), (5.19), the "infinite volume" pressure with wired boundary condition is given by

$$\pi_{\mathbb{G}}(p,q) = -\lim_{N\to\infty} \frac{|\mathbb{E}_N|}{|V_N|} \ln p + \lim_{N\to\infty} \frac{1}{|V_N|} \ln \Psi_N^1(p,q) \tag{5.39}$$

We proved in proposition 4.8 the existence of the first limit in r.h.s. of (5.39), so to prove theorem 5.13 we have to show the existence of the limit

$$\lim_{N\to\infty} \frac{1}{|V_N|} \ln \Psi_N^1(p,q) \tag{5.40}$$

To do this we will use the simpler representation of $\ln \Psi_N^1(p,q)$ in terms of dual animals. So recalling (5.18) we can write

$$\ln \Psi_N^1(p,q) = \sum_{n\geq 1} \frac{1}{n!} \sum_{\mathbf{S}_n\in(\mathcal{E}_N)^n} \Phi^T(\mathbf{S}_n)\rho(\mathbf{S}_n)$$

where again we have used the short notation $\mathbf{S}_n = (S_1, \dots, S_n)$ and $\rho(\mathbf{S}_n) = \rho(S_1) \dots \rho(S_n)$. We also define, for $e \in \mathbb{E}$, the functions

$$\varphi_{\mathbb{G}}(e) = \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{S}_n \in (\mathcal{E}_{\mathbb{G}})^n \\ e \in S_1}} \Phi^T(\mathbf{S}_n) \frac{1}{|S_1|} \rho(\mathbf{S}_n)$$

and

$$F_N = \frac{1}{|V_N|} \sum_{e \in \mathbb{E}_N} \varphi_{\mathbb{G}}(e) \tag{5.41}$$

It is easy to show, by checking the Kotecky-Preiss condition, that the three series above are absolutely convergent as soon as $eA(1+\Delta^{R+1})\varepsilon_p \leq 1$ and hence F_N and $\ln \Psi_N^1(p,q)$ are analytic in δ_p and bounded at least by $C_1\delta_p$ for some constant C_1. Moreover, due to hypothesis that $\mathbb{G}$ is edge quasi-transitive, $\varphi_{\mathbb{G}}(e)$ takes values in a finite set.
Consider now the limit

$$\lim_{N \to \infty} F_N \doteq F_{\mathbb{G}}(q) \tag{5.42}$$

By proposition 4.8 and via an argument completely analogous to that developed in proposition 8 of [35] adapted to edge quasi-transitive graphs, the limit (5.42) exists. Note that to prove the existence of the limit above one needs both vertex transitivity and edge transitivity. Hence, as a limit of bounded analytic functions, $F_{\mathbb{G}}(q)$ is is analytic in p as long as $eA(1+\Delta^{R+1})\varepsilon_p \leq 1$ and bounded by $C_1\delta_p$. This implies that the proof of the theorem is achieved if we show that

$$\lim_{N \to \infty} \frac{1}{|V_N|} \ln \Psi_N^1(p,q) = F_{\mathbb{G}}(q)$$

Observe that

$$\log \Psi_N^1(p,q) - \sum_{e \in \mathbb{E}_N} f_{\mathbb{G}}(e) = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{\mathbf{S}_n \in (\mathcal{E}_N)^n} \Phi^T(\mathbf{S}_n) \rho(\mathbf{S}_n) - \sum_{e \in \mathbb{E}_N} \sum_{\substack{\mathbf{S}_n \in (\mathcal{E}_{\mathbb{G}})^n \\ e \in S_1}} \Phi^T(\mathbf{S}_n) \frac{1}{|S_1|} \rho(\mathbf{S}_n) \right]$$

Now note that

$$\sum_{\substack{\mathbf{S}_n \in (\mathcal{E}_{\mathbb{G}})^n \\ e \in S_1}} (\cdot) = \sum_{\substack{\mathbf{S}_n \in (\mathcal{E}_N)^n \\ e \in S_1}} + \sum_{\substack{\mathbf{S}_n \in (\mathcal{E}_{\mathbb{G}})^n \\ e \in S_1 \\ \exists S_i:\ S_i \not\subset \mathbb{E}_N}} (\cdot)$$

moreover

$$\sum_{e \in \mathbb{E}_N} \sum_{\substack{S_1 \in \mathcal{E}_N \\ e \in S_1}} (\cdot) = \sum_{S_1 \in \mathcal{E}_N} |S_1| (\cdot) \quad , \quad \sum_{e \in \mathbb{E}_N} \sum_{\substack{S_1 \in \mathcal{E}_{\mathbb{G}} \\ e \in S_1}} (\cdot) = \sum_{\substack{S_1 \in \mathcal{E}_{\mathbb{G}} \\ S_1 \cap \mathbb{E}_N \neq \emptyset}} |S_1 \cap \mathbb{E}_N| (\cdot)$$

hence, using also that $|S_1 \cap \mathbb{E}_N|/|S_1| \leq 1$ we get

$$\left| \log \Psi_N^1(p,q) - \sum_{e \in \mathbb{E}_N} \varphi_{\mathbb{G}}(e) \right| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{S}_n \in [\mathcal{E}_{\mathbb{G}}]^n \\ S_1 \cap \mathbb{E}_N \neq \emptyset \\ \exists S_i:\ S_i \not\subset \mathbb{E}_N}} |\Phi^T(\mathbf{S}_n)| |\rho(\mathbf{S}_n)|$$

Let now choose $\ell > R \ln \Delta$ and define

$$m_N^\ell = \frac{1}{\ell} \ln \left[\frac{|V_N|}{|\partial_e V_N|} \right] \tag{5.43}$$

Since by the hypothesis the sequence V_N is Følner, then $\lim_{N\to\infty} m_N^\ell = \infty$, for any $\ell > 0$. We now can rewrite

$$\sum_{\substack{\mathbf{S}_n\in[\mathcal{E}_{\mathbb{G}}]^n\\ S_1\cap\mathbb{E}_N\neq\emptyset\\ \exists S_i:\ S_i\not\subset\mathbb{E}_N}}(\cdot) = \sum_{\substack{\mathbf{S}_n\in[\mathcal{E}_{\mathbb{G}}]^n\\ S_1\cap\mathbb{E}_N\neq\emptyset,\ |\mathbf{S}_n|\geq m_N^\ell\\ \exists S_i:\ S_i\not\subset\mathbb{E}_N}}(\cdot) + \sum_{\substack{\mathbf{S}_n\in[\mathcal{E}_{\mathbb{G}}]^n\\ S_1\cap\mathbb{E}_N\neq\emptyset,\ |\mathbf{S}_n|< m_N^\ell\\ \exists S_i:\ S_i\not\subset\mathbb{E}_N}}(\cdot)$$

Hence

$$\left|\log\Psi_N^1(p,q) - \sum_{e\in\mathbb{E}_N}\varphi_{\mathbb{G}}(x)\right| \leq \sum_{n=1}^\infty\frac{1}{n!}\sum_{\substack{\mathbf{S}_n\in[\mathcal{E}_{\mathbb{G}}]^n\\ S_1\cap\mathbb{E}_N\neq\emptyset,\ |\mathbf{S}_n|\geq m_N^\ell\\ \exists S_i:\ S_i\not\subset\mathbb{E}_N}}\left|\Phi^T(\mathbf{S}_n)\rho(\mathbf{S}_n)\right| +$$

$$+\sum_{n=1}^\infty\frac{1}{n!}\sum_{\substack{\mathbf{S}_n\in[\mathcal{E}_{\mathbb{G}}]^n\\ S_1\cap\mathbb{E}_N\neq\emptyset,\ |\mathbf{S}_n|< m_N^\ell\\ \exists S_i:\ S_i\not\subset\mathbb{E}_N}}\left|\Phi^T(\mathbf{S}_n)\rho(\mathbf{S}_n)\right| \tag{5.44}$$

The first sum can be bounded, for $2e\delta < 1$, by

$$\sum_{n=1}^\infty\frac{1}{n!}\sum_{\substack{\mathbf{S}_n\in[\mathcal{E}_{\mathbb{G}}]^n\\ S_1\cap\mathbb{E}_N\neq\emptyset,\ |\mathbf{S}_n|\geq m_N^\ell\\ \exists S_i:\ S_i\not\subset\mathbb{E}_N}}\left|\Phi^T(\mathbf{S}_n)\rho(\mathbf{S}_n)\right| \leq \text{Const.}|\mathbb{E}_N|\delta_p^{m_N^\ell}$$

which, divided by $|V_N|$, converge to zero as $N\to\infty$ because $|\mathbb{E}_N|/|V_N|$ goes to a constant when $N\to\infty$ (see (4.32)) and by hypothesis $m_N^\ell\to\infty$ as $N\to\infty$.
Concerning the second term in r.h.s. of (5.44), due to the factor $\Phi^T(\mathbf{S}_n)$ the sets S_i must be pair-wise incompatible, which is to say $\cup_i S_i$ must be R-connected. Since $|\cup_i S_i| < \sum_i |S_i| < m_N^p$, from the conditions $S_1\cap\mathbb{E}_N\neq\emptyset$ and $S_i\not\subset\mathbb{E}_N$, we conclude that all polymers S_i must lie in the set

$$\mathrm{B}^e_{m_N^\ell}(\partial V_N) = \{e\in\mathbb{E} : \frac{1}{R}d_{\mathbb{G}}(e,\partial_e V_N)\leq m_N^\ell\}$$

with cardinality bounded by

$$|\mathrm{B}_{m_N^p}(\partial V_N)| \leq |\partial_e V_N|\Delta^{Rm_N^p+1}$$

Hence we have that second sum in r.h.s. of (5.44) is bounded by

$$\sum_{n=1}^\infty\frac{1}{n!}\sum_{\substack{\mathbf{S}_n\in[\mathcal{E}_{\mathbb{G}}]^n\\ S_1\cap\mathbb{E}_N\neq\emptyset,\ |\mathbf{S}_n|< m_N^\ell\\ \exists S_i:\ S_i\not\subset\mathbb{E}_N}}\left|\Phi^T(\mathbf{S}_n)\rho(\mathbf{S}_n)\right| \leq \text{Const}'.|\partial_e V_N|\Delta^{Rm_N^\ell}\delta$$

Thus recalling definitions (5.41) and (5.43), we have

$$\left|\frac{1}{|V_N|}\log\Psi_N^1(p,q) - \frac{1}{|V_N|}\sum_{x\in V_N}\varphi_{\mathbb{G}}(x)\right| = \left|\frac{1}{|V_N|}\log\Xi_{\mathbb{G}|_{V_N}} - F_{\mathbb{G}}(q)\right| \leq$$

$$\leq \text{Const.}\frac{|\mathbb{E}_N|}{|V_N|}\delta_p^{m_N^\ell} + \text{Const}'.\frac{|\partial_e V_N|}{|V_N|}\Delta^{Rm_N^\ell}\delta$$

$$\le \text{Const.} \left[\frac{|\partial V_N|}{|V_N|}\right]^{\frac{|\ln \delta_p)|}{\ell}} + \text{Const.}\delta \left[\frac{|\partial V_N|}{|V_N|}\right]^{1-\frac{R\ln\Delta}{\ell}}$$

Since by hypothesis $|\partial V_N|/|V_N| \to 0$ as $N \to \infty$, we conclude that the quantity above is as small as we please for N large enough. This ends the proof of the theorem. $\Box$

Acknowledgements.

We thank an anonymous referee for his careful work of revision and for many useful comments, remarks and suggestions which helped to improve the quality of the paper. A.P. also thanks the brazilian agencies CAPES, CNPq and FAPEMIG for financial support.

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